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MODELING OF ATTRITION IN A
BARRIER PENETRATION PROCESS WITH
EVENLY SPACED BARRIER ELEMENTS

Alan F. Karr
Eleanor L. Schwartz

September 1990

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INSTITUTE FOR DEFENSE ANALYSES

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PREFACE

This paper was prepared under the Institute for Defense Analyses' Central Research Program. It is a derivation and expansion of some attrition equations incorporated into IDA's NAVMOD model of naval combat (Reference [3]).

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ABSTRACT

This paper examines a class of combat processes where one side's resources are evenly distributed along a barrier and the other side attempts to penetrate that barrier. Starting from a set of straightforward assumptions, the paper derives exact or approximate expressions for the expected numbers of resources destroyed, the expected number of shots fired at a target, and in some cases, the probability distribution of the number of shots fired at a target. The derivations combine and extend several concepts used in previous attrition models, including having resources situated on a barrier, letting each attacker engage several targets, and the use of the Poisson approximation to the binomial when there are several types of targets. Several different firing rules (attack protocols) are considered. Some of the equations developed here have been used in the Institute for Defense Analyses' NAVMOD naval model to model attrition when a task force or group of ships crosses a hostile submarine barrier.

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INTRODUCTION

This paper examines a class of combat processes where one side's resources are evenly distributed along a barrier and the other side attempts to penetrate that barrier. Starting from a set of straightforward assumptions, the paper derives exact or approximate expressions for the expected numbers of resources destroyed, the expected number of shots fired at a target, and, in some cases, the probability distribution of the number of shots fired at a target. The derivations combine and extend several concepts used in previous attrition models [5, 6, 8, 9, 10]. Some of these concepts are 1) having resources situated on a barrier, 2) letting each attacker engage several targets, and 3) the use of the Poisson approximation to the binomial when there are several types of targets. Several different attack protocols (firing rules) are considered.

Some of the equations developed here have been used in the NAVMOD naval model [3] to model attrition when a task force or group of ships crosses a hostile submarine barrier. However, the barrier could be composed of any type of resource (e.g., aircraft) spaced evenly along a barrier line. Accordingly, in the rest of the paper we shall speak of "barrier elements" and "penetrators" with the understanding that these could possibly be ships, submarines, aircraft or any appropriate types of resources. In the processes examined here, there can be several types of penetrators but only one type of barrier element. (Strictly speaking, there can be several types of barrier elements present as long as the effectiveness parameters considered are the same for all the types.)

Chapter I considers the cases where the barrier elements are the shooters and the penetrators are the targets. In Chapter II, the penetrators are the shooters and the barrier elements, the targets. The issue of combining two one-sided attrition processes into a process where attrition to both sides is computed is not addressed in this paper. Reference [1] gives one method for doing this; Reference [7] gives some justifications for this method. This paper is structured so that in Chapter I, Section G, which contains results, can be read immediately after Section A, the statement of the problem, without loss of continuity. Similarly, in Chapter II, Section E can be read after Section A without loss of

continuity. Appendices A and B provide proof details of the results presented in Chapters I and II, respectively.

The following notation is used throughout the paper. If A and B are events, then $P(A)$ denotes the probability of the event A and AB denotes the intersection of the events A and B . A random variable is denoted by a symbol with a tilde over it, a realized value of such a random variable by the corresponding symbol without the tilde. Thus the random variable \tilde{h} might assume a value h ; its expectation is $E(\tilde{h})$ and for a given value h , one can speak of the probability $P(\tilde{h}=h)$. The symbol "l" should be read as a lower case "ell." For any nonnegative real x , $\lfloor x \rfloor$ denotes the integer part of x , i.e., the greatest integer less than or equal to x , and $\langle x \rangle$ denotes the fractional part of x , i.e., $\langle x \rangle = x - \lfloor x \rfloor$.

I. PENETRATORS ARE TARGETS, BARRIER ELEMENTS ARE ATTACKERS

A. THE PROBLEM

Suppose that there are n types of penetrators (n could equal 1) with R_j penetrators of type j , for $j=1,\dots,n$. Let

$$R = \sum_{j=1}^n R_j$$

denote the total number of penetrators. There is one type of barrier element; there are B barrier elements present. We consider five attrition processes, indexed by attack protocol (a) through (e) in Assumption 5) below, that proceed according to the following assumptions.

- 1) Barrier elements are spaced evenly along a barrier of length L , i.e., the line of length L is divided into B equal segments and a barrier element is positioned at the center of each segment.
- 2) At a given time, each penetrator picks a crossing point \tilde{x} along the barrier line, according to a uniform distribution, and all penetrators attempt to cross the barrier simultaneously.
- 3) The crossing points of different penetrators are mutually independent random variables.
- 4) If the crossing point of a given penetrator of type j lies within distance $w_j/2$ of a given barrier element, that barrier element will detect the penetrator with probability d_j . Otherwise, that barrier element will not detect the penetrator. Each barrier element detects the penetrators that come within $w_j/2$ of it independently of one another.
- 5) Of the penetrators it has detected, each barrier element chooses certain ones to attack with certain numbers of shots (salvos), according to one of the attack protocols described below. In protocols (b), (c), and (d), the parameter H is that number of salvos available per barrier element. (All barrier elements use the same attack protocol.)
 - (a) A barrier element fires exactly one salvo at each penetrator it detects.

- (b) Of the penetrators it has detected, each barrier element chooses one according to a uniform distribution and fires one salvo at it. The barrier element performs this process exactly H times. The target choices for successive firings are independent.
- (c) If a barrier element detects m targets (penetrators), it fires $\lfloor H/m \rfloor$ salvos at each target, then chooses according to a uniform distribution $H - m\lfloor H/m \rfloor$ of the m targets and fires one additional salvo at each.
- (d) If the number of targets, m , that a barrier element detects does not exceed H , the shooter fires exactly one salvo at each target. If $m > H$, the shooter chooses according to a uniform distribution H of the m targets and fires one salvo at each.
- (e) Of the targets a barrier element has detected, it chooses one target according to a uniform distribution and fires exactly one salvo at it. (This is the special case of protocols (b), (c), and (d) when $H = 1$. We consider it as a separate protocol because the formulas for the desired quantities are considerably simpler.)
- (6) Conditional on detection of that penetrator, the decisions of different barrier elements to attack a given penetrator are independent events.
- (7) A shot (salvo) fired at a penetrator of type j kills it with probability k_j .
The effects of different shots (fired at the same penetrator) are independent.

We wish to compute the following quantities:

- The expected number of penetrators of type j killed.
($j = 1, \dots, n$)
- The expected number of shots (salvos) fired at a given penetrator.
- The probability distribution of the number of shots fired at a given penetrator.

The probabilistic derivations of the equations assume that integer numbers of combatants are present. Strictly speaking, therefore, B and all the R_j should be integers. In the formulas which have a closed form, however, it probably does little harm to use nonintegral values for B and the R_j , if the values are at least 1. (In a deterministic combat model, the iterative use of attrition equations can yield noninteger numbers of combatants, so attrition formulas that can be evaluated for noninteger numbers of resources are desirable.) Where a formula is expressed as an indicated sum with a limit of R or $R-1$, interpolation between the results with $\lfloor R \rfloor$ and $\lfloor R \rfloor + 1$ is a reasonable procedure. At least one of the R_j should be nonzero, and B should be nonzero.

In some formulas, the number of possible shots per shooter H , which is used in attack protocols (b), (c), and (d), is the limit of an indicated sum, and thus should be a positive integer--but reasonable values for H might frequently be such. (Throughout this paper the terms engagement, shot, and salvo will be used synonymously.) Reliability of shots can be considered in determining the values of the kill probability k_j . The barrier length L should be strictly positive, and the "detectability width" w_j should be less than or equal to L , for all j . (In the "homogeneous" case, i.e., $n=1$, w , d , and k are used instead of w_j , d_j , and k_j , respectively.)

Exact or approximate formulas for the three main quantities of interest have been developed for each attack protocol, although in some cases, it has been necessary to specify further restrictions on the parameter values, as discussed in Section D. There are two reasons why some of the formulas are only approximations. The first arises from the use of the Poisson approximation to the binomial. To develop algebraically tractable formulas, we have assumed in some cases that certain binomial random variables are approximately Poisson distributed. Section E explains where in the derivations these assumptions are made.

The second source of approximation is the "ignoring of edge effects." This paper derives formulas for the desired quantities in terms of the probabilities of certain events that involve a single penetrator and/or a single barrier element. The paper posits certain expressions for these probabilities that are reasonable, and are exact in certain cases, but might not be exact if a barrier element is located near the edge of the barrier, or if the crossing point of a penetrator is near the edge of the barrier (where "near" depends on the values of certain parameters). These expressions, and the formulas developed from them, thus ignore edge effects.¹ Section F discusses some of these expressions and the reasons why they are not exact. Other such expressions are explained in relevant parts of the paper.

Without loss of continuity, the reader can skip to Section G, which presents the formulas that have been developed for the three quantities of interest. Each theorem in Section G states if edge effects have been ignored in its derivation, and states restrictions on the parameters as necessary. The formulas are stated as approximate equalities if the Poisson approximation to the binomial has been used in the derivation.

¹ Exact expressions for the relevant probabilities would have to take the edge effects into account, and might be more complicated than the expressions used here.

B. DISCUSSION OF ASSUMPTIONS

The first assumption, the geometric placement of the barrier elements, is the feature that distinguishes this paper from certain other work on attrition equations. (Strictly speaking, for the derivations in this chapter to hold, it is not necessary to assume that a barrier element is always positioned at the center of its segment. It is merely necessary that the barrier element can potentially detect and attack all type-j penetrators--and only those type-j penetrators--that transit within distance $w_j/2$ of the center point.) Another barrier penetration model [2] assumes somewhat more independence between the barrier elements.

The second and third assumptions are a simplification of reality: a group of penetrators might cross a barrier in some coordinated fashion, or sequentially. In the NAVMOD model, some of the formulas developed here are used to model barrier crossings, but provisions are made for the penetrating ships to cross along a "corridor width" that may be less than the full physical barrier length L . The details appear in [3]. Note that the uniform distribution of the penetrators' crossing points implies that a penetrator cannot take advantage of knowledge of the barrier elements' positions. In particular, penetrators cannot discern or exploit gaps in the barrier coverage.

Throughout this chapter we will use the following terminology. If the crossing point x of a particular penetrator of type j , penetrator p , is within distance $w_j/2$ of the location of barrier element β (which can be thought of as the midpoint of the β th interval of length L/B from one end of the barrier) we will use the phraseology " p is detectable by β ," or " p is vulnerable to β ", or " β can potentially detect p ." If β can potentially detect p , then β detects p (" β actually detects p ," " p is detected by β ") with probability d_j . The "detectability width" w_j could be thought of as a sweepwidth against type- j penetrators (especially if $d_j = 1$). It is possible, if the w_j are large, that the detectability areas of different barrier elements overlap, i.e., that a penetrator is vulnerable to more than one barrier element. Frequently, however, it is reasonable to set the parameter w_j to L/B for all j . The interpretation of this is that each barrier element is assigned its own "barrier cell," and can attack only those penetrators that come within its cell. We have tacitly assumed that a barrier element is capable of attacking any penetrator it detects. Therefore, w_j should not be set so large that a barrier element detects penetrators that are too far away to attack.

The second part of assumption 4) can be stated more precisely as follows. Given any specific barrier element β and penetrators (of whatever types) p_1, p_2, \dots, p_k that are detectable by β , let D_i denote the event that β actually detects penetrator p_i . The second

part of assumption 4) states that the D_i form a set of mutually independent events. This assumption is of key importance in the derivations of the formulas.

We note that in assumption 5), the choices of target(s) to be attacked do not depend on the types of penetrators detected by a barrier element, but only on the total number of penetrators detected. In general, deriving attrition formulas where the target choice is dependent on target type is a difficult probabilistic problem. For a solution of one special case of this, see Reference [11].

If the detectability widths for different barrier elements (the portion of the barrier with $w_j/2$ on either side of a barrier element) overlap (this could happen if w_j or B is large), it is possible that two or more barrier elements can detect the same penetrator. Assumption 6) merely states that those barrier elements do not coordinate their attacks. This is not unreasonable if one assumes there is no communication between barrier elements (e.g., barrier submarines). Assumption 7) has been used in most of the IDA work on attrition equations [5,9]; it implies that if a penetrator of type j receives exactly h shots, it will survive with probability $(1-k_j)^h$. This is a realistic assumption in the case where a fired shot is lethal if it hits the target, but has some probability (due, say to unreliability) that it does not hit, and the reliabilities of different shots are independent. On the other hand, it may be that multiple shots hitting a target have a cumulative effect. For example, there is the situation where one hit does no damage at all to a target, but two hits destroy it. To model a situation with this property, it is often desirable to know the probability distribution of the number of shots fired at a target. In this paper, we derive this probability distribution for some cases. Thus, in computing attrition, assumption 7) could be relaxed and a function giving the probability a target is destroyed if h shots are fired at it could be used instead. NAVMOD [3] has provisions for modeling situations where hits have a cumulative effect.

We note finally that the case of multiple penetrator types-- $n > 1$ -- is essentially intractable if the exact formulas are desired. In this paper we derive the exact formulas for the case where $n = 1$ and approximations for the multiple penetrator type case. This will be discussed more in Section E below.

C. RELATION TO PREVIOUS WORK

The attrition equations developed in this paper are part of a series of "binomial" attrition models. Such models can be considered as discrete time analogs of Lanchester attrition equations. Initial numbers of searchers and targets and certain effectiveness

parameters are input. Attrition occurs over an interval of time, according to specified assumptions concerning detection, attack, and kill. Formulas for the expected number of targets killed (and, in some cases, other quantities of interest) are probabilistically derived from the assumptions. The initial work at IDA on binomial attrition processes appears in Reference [5]. Reference [10] derives the equations for many such processes; Reference [9] presents results for most of the important processes. Derivations of equations for certain specific binomial attrition processes appear in [6], [11], [12], and [13].

The main difference between this paper and the above references is the explicit positioning of the resources along a barrier line. The previously developed binomial attrition models considered uncoordinated forces of attackers: different attackers were mutually independent. Here, because of the barrier spacing, different attackers (barrier elements) are not independent. (Different penetrators are independent.) To derive expected kills here, we condition on the crossing point of a penetrator and then still must ensure that certain random variables are conditionally independent given this crossing point. This line of argument was unnecessary in previous work. We point out that attack protocol (a) (each attacker shoots once at each target it detects) was trivial to treat when attackers were uncoordinated. Introducing the barrier structure makes protocol (a) more interesting to examine.

This paper also treats in an integrated fashion the issues of multiple shots per shooter and multiple target types. Of the references discussed here, only Reference [6] allowed attackers to have several shots: attack protocol (d) was used and only one type of target (and one type of shooter) was considered. The attrition equation developed in [6] (which is also reported in [9]) does not have a closed form: it is expressed in terms of cumulative binomial probabilities. The same is true of most of the equations developed in the current paper for attack protocols (b), (c), and (d), where an attacker has more than one shot available.

Binomial attrition processes involving targets (and shooters) of more than one type have been explored much more fully. If detection parameters are a function of target type, the expected number of targets of type j killed can be rigorously derived, but the formula is essentially intractable algebraically [5]. To circumvent this difficulty, the Poisson approximation to the binomial has been used to develop approximate attrition equations that have simpler forms [8]. In some cases, error bounds on the approximation accuracy of the final attrition equation have been developed [8, 10]. (Kill parameters can, without difficulty, be functions of both shooter type and target type. Reference [12] examines the

process where detection parameters are functions of shooter type but not target type. The resulting attrition equation, first derived in [5], has a relatively simple closed form.)

Two other models that consider combat processes similar to the ones treated in this paper are "An Attrition Model for Penetration Processes" [2], and the attrition routines of the "Barrier Air Defense Model" [4, especially pp. E-43ff]. Reference [2] examines a combat process in which penetrators attempt to get through a set of defenders, that can be considered as forming a barrier. Probability distributions—not just expected values—of a wide variety of quantities are computed. The assumptions of [2] are somewhat different from those of this paper: penetrators attack one by one, in succession; the attack can result in attrition to both penetrators and defenders, and each defender detects a penetrator independently of other defenders. Reference [2] admits that there are situations where this latter assumption is not realistic, particularly if the defenders are evenly spaced barrier elements.

The Barrier Air Defense Model considers a group of evenly spaced penetrating aircraft opposed by defending aircraft. Several combat processes are considered; in one, the defending aircraft are also evenly spaced, in others, the defending aircraft are positioned to maximize expected attrition to the penetrators.¹ A defender can detect only those penetrators that come within the defender's "launch opportunity zone"; detected penetrators can be attacked according to a number of protocols similar to those in Assumption 5 (of this paper). The assumption of evenly spaced (rather than independent random) penetrators distinguishes Reference [4] from the work reported here. For cases where penetrators tend to cluster when crossing a hostile barrier, the assumption of independent random penetrators provides a tractable way of examining the effects of such clustering. A clustered crossing pattern is not the same as independent random penetrators, but the latter case is tractable mathematically and is in some sense conservative: no coordination or communication between different penetrators is assumed. The results can thus provide a benchmark for evaluating the effectiveness for the penetrating side of any particular crossing pattern.

Reference [14] examines a penetration process that involves a single transiting ship crossing a minefield. Each mine is placed along a barrier line according to a uniform distribution; the locations of the mines are independent of one another. The crossing point

¹ In cases where the barrier must be formed before the positions of the penetrators are known, even spacing of barrier elements along the barrier line would seem to be the more appropriate assumption.

of the penetrator is uniformly distributed along the barrier line and is independent of the positions of the mines. The probability that the transitor is killed is derived, and the influence of edge effects is thoroughly examined (cf. Section F of this chapter).

D. AN INDEPENDENCE CONDITION

We have been able to derive formulas for certain of the desired quantities only if it can be guaranteed that certain random variables are independent or conditionally independent. In addition to the basic assumptions 1) through 7), certain constraints on the parameters are sometimes necessary. To explain the reasons for this, let us examine how one might go about deriving the formula for the expected number of penetrators of type j killed.

This quantity is the number of penetrators of type j multiplied by the probability that a specific penetrator of type j is killed by the barrier elements [12]. In symbols,

$$E(\Delta \tilde{R}_j) = R_j(1-P(S)) ,$$

where the random variable $\Delta \tilde{R}_j$ is the number of penetrators of type j killed and S is the event that a given penetrator (call it penetrator ρ) of type j survives the barrier. We thus wish to compute $P(S)$. Let the specific barrier elements be indexed by b , ranging from 1 through B . For each b , let S_b be the event that penetrator ρ survives barrier element b . The intersection of the events S_b is S , but the individual S_b are clearly not independent: if penetrator ρ survives barrier element β , the chance is increased that ρ 's crossing point was not within the detectability area of β , thus the crossing point is more likely to be in the detectability area of some other barrier element (call it γ), and thus the probability that ρ has survived γ decreases. The interesting question is whether the events S_b are conditionally independent given the crossing point x of penetrator ρ .

Consider the case where the detectability widths of the barrier elements overlap (i.e., $w_j > L/B$), and where the attack protocol is (b), (c), (d), or (e), so that the number of shots available per barrier element is limited. Suppose that penetrator ρ 's given crossing point x is such that ρ is vulnerable to several barrier elements (call them $\beta_1, \beta_2, \dots, \beta_k$). The probability that barrier element β_i fires at ρ depends, in part, on the number of other penetrators barrier element β_i detects. In any realization of the process, the crossing points of the other penetrators are fixed. If the penetrator survives barrier element β_1 , this might indicate that fewer shots than average were fired at ρ by β_1 --which might have happened because β_1 detected many penetrators and had to spread its fixed number of shots thinly. This would imply that many penetrators crossed within the detectability area of β_1 , which

would indicate that the number of penetrators crossing within the detectability area of some other barrier element, say β_2 , was less than average. Since p is vulnerable to β_2 , the chance of p receiving more shots from β_2 , hence being killed, would be increased.¹

The above argument merely indicates why it is not obvious that the events S_b are conditionally independent given the crossing point x of penetrator p ; it does not imply that the S_b are not conditionally independent. Unfortunately, numerical examples have been developed that show that the events S_b are indeed not necessarily conditionally independent given x . In this case, $P(S)$ must be found by integrating over the appropriate joint probabilities of the specific crossing points of all penetrators, an involved calculation that we have not attempted in this paper.

Under certain conditions, however, the S_b do form a set of conditionally independent events. Specifically, with p , x , and S_b defined as above, the following two results hold. (Proofs are in Appendix A.)

Lemma I.1: If attack protocol (a) is used (each barrier element fires exactly one shot at each penetrator it detects) then the S_b are conditionally independent given x .

Lemma I.2: If penetrator p is of type j and $w_j \leq L/B$, then the S_b are conditionally independent given x . This is true whether or not $w_i \leq L/B$ for other target types i .

One reasonable scenario where the premises of Lemma I.2 are met occurs when the barrier line (of length L) is partitioned into B equal nonoverlapping "cells" and a barrier element is allowed to attack only those penetrators it detects whose crossing points lie within its cell. Then, for all j , the parameter w_j is the minimum of the "true effective sweep width" and L/B .

If protocol (a) is used or if $w_j \leq L/B$, we will use the terminology "the independence condition holds." The independence condition is used in the derivations of the expected number of penetrators killed and the probability distribution of the number of shots fired at a penetrator. The formulas for the expected number of shots fired at a penetrator, however, are valid whether or not the independence condition holds.

¹ This argument is similiar in spirit, though not in detail, to the explanation of why targets do not die independently of one another, given in Reference [5], p. 5.

E. DERIVATION OF THE DESIRED QUANTITIES

This section explains the main arguments used in the derivation of the three desired quantities, presenting the formulas when attack protocol (b) is used. Proof details appear in Appendix A. The homogeneous ($n = 1$) and heterogeneous ($n > 1$) cases are considered separately, where appropriate. In the course of the derivation, we point out certain issues of approximation, which are discussed more fully in Section F.

We start the derivation by considering one specific penetrator, penetrator ρ , which is assumed to be of type j , and one specific barrier element β *that can potentially detect* ρ . We then use the results of this situation to derive the overall quantities of interest.

1. One Penetrator, One Barrier Element

We assume that the crossing point x of penetrator ρ is such that barrier element β can potentially detect ρ --i.e., x is within $w_j/2$ of the location of β (which is at the midpoint of the β th interval of length L/B along the barrier line). We define the following events and random variables:

D-- β actually detects ρ

A_1 --The number of penetrators other than penetrator ρ (regardless of type), that β detects, is exactly 1. (Defined for $i=0,1,\dots,R-1$.)

DA_1 --The intersection of events D and A_1 , i.e., β detects ρ and exactly 1 other penetrators.

\tilde{h} --random variable indicating the number of shots β fires at ρ .

F_h --The event that $\tilde{h}=h$, i.e., β fires exactly h shots at ρ . (Defined for $h=0,1,\dots,H$.)

K-- β kills ρ .

We compute $P(K)$, $E(\tilde{h})$, and the probabilities $P(F_h)$. These correspond to our three desired quantities, but in the case where only one barrier element and one penetrator are present. Later, we integrate these quantities into overall results. Before we derive $P(K)$, $E(\tilde{h})$, and the $P(F_h)$ we derive their "component parts" $P(F_h | DA_1)$, $P(K | F_h)$, and $P(DA_1)$.

Because the choice of targets β attacks depends only on the number, not the types, of penetrators detected, the conditional probabilities $P(F_h | DA_1)$ are functions of h and i only. If attack protocol (b) is used, from the statement of Assumption 5b) it is clear that, if

β detects ρ and ι other targets, i.e., $\iota+1$ targets in total, the number of shots target ρ receives is binomially distributed with parameters H and $1/(\iota+1)$. In symbols,

$$P(F_h | DA_\iota) = \binom{H}{h} \left(\frac{1}{\iota+1} \right)^h \left(\frac{\iota}{\iota+1} \right)^{H-h}.$$

This is defined for $h=0, \dots, H$ and $\iota=0, \dots, R-1$, and in fact is sensible mathematically for any nonnegative ι . (Formulas for $P(F_h | DA_\iota)$ for the other attack protocols appear and are derived in Appendix A [Lemma I.5].)

Let a bar over any event denote its complement, so \bar{K} is the event that β does not kill ρ . From Assumption 7), it is clear that

$$P(\bar{K} | F_h) = (1-k_j)^h,$$

so

$$P(K | F_h) = 1 - (1-k_j)^h, \text{ for every } h.$$

The probabilities $P(DA_\iota)$ are somewhat more interesting. Since penetrator ρ is assumed to be detectable by β , $P(D)$ is simply d_j . Consider some other penetrator, ρ' , which is of type i . In order to be detected by β , penetrator ρ' must first be detectable by β --that is, its crossing point must be within $w_i/2$ on either side of the location of β . *Ignoring edge effects*, the probability of this is, by Assumption 2), $2(w_i/2)/L$, or w_i/L . Given that ρ' is detectable by β , ρ' will actually be detected by β with probability d_i . Thus

$$P(\text{penetrator } \rho', \text{ of type } i, \text{ is detected by } \beta) = d_i w_i / L.$$

The potential edge effects problems will be discussed in Section F, below. By Assumption 3), crossing points of different penetrators are independent, and by the second part of Assumption 4), penetrators detectable by barrier element β are actually detected by β independently of one another. Thus, if $\rho_1, \rho_2, \dots, \rho_m$ are different penetrators, the events

$$(\beta \text{ actually detects penetrator } \rho_k)$$

for $k=1, \dots, m$ form a mutually independent set. By this same reasoning, the event D (β detects ρ) is independent of the event β detects ρ_k , for $\rho_k \neq \rho$. There are $R - 1$ penetrators other than penetrator ρ . Therefore, *if there is only one type of penetrator*, the number of penetrators other than penetrator ρ that barrier element β detects is binomially distributed with parameters $R-1$ and wd/L . In symbols,

$$P(A_t) = \binom{R-1}{t} \left(\frac{wd}{L} \right)^t \left(1 - \frac{wd}{L} \right)^{R-1-t}$$

and

$$P(DA_t) = dP(A_t) = d \binom{R-1}{t} \left(\frac{wd}{L} \right)^t \left(1 - \frac{wd}{L} \right)^{R-1-t},$$

for $t=0, \dots, R-1$.

Now suppose that $n > 1$. (The following argument is identical to that given in Reference [9].) Recall that the specific penetrator ρ under consideration is of type j . Not including penetrator ρ , there are $R_j - 1$ penetrators of type j and R_i penetrators of type i , for all $i=1, \dots, n$, $i \neq j$. Let the random variable \tilde{t}_i denote the number of type- i penetrators, other than penetrator ρ , detected by β . It is clear from the preceding arguments that:

- For $i \neq j$, \tilde{t}_i is distributed binomially with parameters R_i and $w_i d_i / L$.
- \tilde{t}_j is distributed binomially with parameters $R_j - 1$ and $w_j d_j / L$.
- $\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n$ are a mutually independent set of random variables.

The sum

$$\tilde{t} = \sum_{i=1}^n \tilde{t}_i$$

is the total number of penetrators other than penetrator ρ that β detects. The distribution of \tilde{t} is a convolution of binomial distributions with different parameters, and the resulting formula is algebraically intractable [5]. We therefore use the Poisson approximation to the binomial and consider each \tilde{t}_i as being approximately Poisson distributed with mean $R_i w_i d_i / L$, for $i \neq j$. Random variable \tilde{t}_j is approximately Poisson with mean $(R_j - 1) w_j d_j / L$. Since the sum of independent Poisson random variables is Poisson, \tilde{t} is approximately distributed Poisson with mean

$$\alpha_j = (R_j - 1) w_j d_j / L + \sum_{\substack{i=1 \\ i \neq j}}^n R_i w_i d_i / L.$$

Then the probability $P(A_t) = P(\tilde{t}=t)$, i.e., the probability that barrier element β detects exactly t targets other than penetrator ρ is approximately

$$\alpha_j^1 e^{-\alpha_j} / 1! ,$$

and

$$P(DA_i) \approx d_j \frac{\alpha_j^1 e^{-\alpha_j}}{1!} .$$

We now return to the derivation of $P(K)$, $E(\tilde{h})$, and $P(F_h)$. The events F_0 through F_H form a partition, thus

$$P(K) = \sum_{h=0}^H P(K|F_h) P(F_h) = \sum_{h=1}^H P(K|F_h) P(F_h),$$

since if β fires zero shots at ρ , K will not occur. Similarly

$$E(\tilde{h}) = \sum_{h=1}^H h P(F_h) .$$

The probability that penetrator ρ receives exactly h shots from barrier element β depends on the total number of targets β has detected. The events \bar{D} , DA_0 , DA_1, \dots, DA_{R-1} form a partition, so

$$P(F_h) = P(F_h|\bar{D}) P(\bar{D}) + \sum_{i=0}^{R-1} P(F_h|DA_i) P(DA_i) .$$

We have assumed that no shot fired by a barrier element can reach a penetrator not detected by that barrier element. Thus $P(F_0|\bar{D}) = 1$ and $P(F_h|\bar{D}) = 0$ for all $h \geq 1$. Therefore

$$P(K) = \sum_{h=1}^H \sum_{i=0}^{R-1} P(K|F_h) P(F_h|DA_i) P(DA_i)$$

and

$$E(\tilde{h}) = \sum_{h=1}^H \sum_{i=0}^{R-1} h P(F_h|DA_i) P(DA_i) .$$

The appropriate formulas derived earlier can be substituted for the indicated probabilities, and the algebra performed. The following points should be noted. First, it is generally easier to compute $P(K)$ and $E(\tilde{h})$ by interchanging the order of summation and summing on h first. The formulas for $P(F_h)$ are, in general, complicated algebraically. Second, many of the formulas do not reduce to a closed form and must be left as indicated summations. Third, in the heterogeneous case ($n > 1$) the approximate (Poisson) formulas for $P(DA_i)$ will be used. To compensate for the Poisson approximation to the binomial,

indicated sums on ι will be taken over all nonnegative values of ι , not just up to $R-1$. (Note that the formulas for both $P(F_h|DA_\iota)$ and the approximation to $P(DA_\iota)$ are well defined for all nonnegative integer ι , and Fubini's theorem justifies interchanging the order of summation on h and ι .)

In attack protocol (b), most of the formulas will not reduce to closed forms. Let us introduce the following alternative notation for $P(DA_\iota)$. In the homogeneous case (one type of penetrator), we denote $P(DA_\iota)$ by ω_ι , where

$$\omega_\iota = d \binom{R-1}{\iota} \left(\frac{wd}{L} \right)^\iota \left(1 - \frac{wd}{L} \right)^{R-1-\iota} \quad \iota=0,\dots,R-1$$

If $n > 1$, we first define

$$\alpha_j = (R_j - 1) \frac{w_j d_j}{L} + \sum_{\substack{i=1 \\ i \neq j}}^n R_i \frac{w_i d_i}{L}$$

and then the Poisson approximation to $P(DA_\iota)$ by $\psi_{j\iota}$, where

$$\psi_{j\iota} = d_j e^{-\alpha_j} \alpha_j^\iota / \iota! \quad \iota = 0, 1, \dots$$

This assumes that the specific penetrator ρ under consideration is of type j . (A further approximation, which is somewhat easier computationally, is to replace α_j by

$$\bar{\alpha} = \sum_{i=1}^n R_i \frac{d_i w_i}{L}$$

in each $\psi_{j\iota}$. Reference [8] does this; here we will leave the formulas in terms of the α_j .)

Using this notation, after algebraic simplification, the formulas for the quantities $P(K)$, $E(\tilde{h})$, and $P(F_h)$, where attack protocol (b) is used, are as follows. In the homogeneous case

$$P(K) = d - \sum_{\iota=0}^{R-1} \omega_\iota \left(\frac{\iota+1-k}{\iota+1} \right)^\iota,$$

$$E(\tilde{h}) = \frac{LH}{wR} \left[1 - \left(1 - \frac{wd}{L} \right)^R \right],$$

$$P(F_0) = 1-d + \sum_{i=0}^{R-1} \omega_i \left(\frac{1}{i+1} \right)^H,$$

and

$$P(F_h) = \sum_{i=0}^{R-1} \omega_i \binom{H}{h} \left(\frac{1}{i+1} \right)^h \left(\frac{1}{i+1} \right)^{H-h}, \quad h=1, \dots, H.$$

In the heterogeneous ($n > 1$) case, assuming that the specific penetrator is of type j , the following approximate formulas obtain:

$$P(K) \approx d_j - \sum_{i=0}^{\infty} \psi_{ji} \left(\frac{1+i-k_j}{i+1} \right)^H,$$

$$E(\tilde{h}) \approx \frac{d_j H}{\alpha_j} (1 - e^{-\alpha_j}),$$

$$P(F_0) \approx 1 - d_j + \sum_{i=0}^{\infty} \psi_{ji} \left(\frac{1}{i+1} \right)^H,$$

and

$$P(F_h) \approx \sum_{i=0}^{\infty} \psi_{ji} \binom{H}{h} \left(\frac{1}{i+1} \right)^h \left(\frac{1}{i+1} \right)^{H-h}, \quad h=1, \dots, H.$$

2. Combined Results Considering All Barrier Elements

The previous subsection considered one penetrator ρ , of type j , and one barrier element β that could potentially detect ρ . Using the quantities $P(K)$, $E(\tilde{h})$, and the $P(F_h)$ derived in Subsection E.1, we now derive the overall quantities of interest. We again consider one penetrator ρ , assumed to be of type j . The individual barrier elements are indexed by b , $b=1, \dots, B$. We use the following notation, some of which was introduced in Section D.

S_b -- event that barrier element b does not kill penetrator ρ (i.e., ρ survives b).

$S = \bigcap_{b=1}^B S_b$, i.e., the event that ρ survives all the barrier elements.

\tilde{y}_b -- random variable indicating the number of shots fired at ρ by barrier element b .

$\tilde{y} \rightarrow \sum_{b=1}^B \tilde{y}_b$ i.e., the total number of shots fired at ρ .

\tilde{x} --point on the barrier at which penetrator ρ crosses, uniformly distributed on $[0, L]$.

C_x --event that $\tilde{x} = x$, defined for $x \in [0, L]$.

$N_j(x)$ --number of barrier elements that can potentially detect ρ , if ρ is of type j and the crossing point of ρ is x . If $n=1$ we will use the notation $N(x)$.

Again, the quantities we wish to derive are:

- The expected number of penetrators of type j killed.
- The expected number of shots fired at a given penetrator.
- The probability distribution of the number of shots fired at a given penetrator.

In terms of the above notation, these desired quantities are given by $R_j(1-P(S))$, $E(\tilde{y})$ and $P(\tilde{y}=y)$ (for appropriate y), respectively. We derive each quantity in turn.

To find $P(S)$ we first condition on the crossing point x . We shall speak of probabilities conditional on C_x , even though $P(C_x)$ is zero for every specific x . The crossing point \tilde{x} is distributed uniformly on $[0, L]$, thus

$$P(S) = \int_0^L P(S|C_x) \frac{1}{L} dx.$$

If the independence condition holds, as described in Section D, then the events S_b are conditionally independent given x , so that

$$P(S|C_x) = \prod_{b=1}^B P(S_b|C_x), \quad x \in [0, L].$$

If x is not within the detectability area of barrier element b , the penetrator will certainly survive barrier element b , so $P(S_b|C_x) = 1$; if x is within the detectability area of b , then by the results of the previous section, $P(S_b|C_x) = 1 - P(K)$, regardless of the specific barrier element b involved. Thus

$$\prod_{b=1}^B P(S_b|C_x) = (1-P(K))^{N_j(x)}.$$

The geometric spacing of the barrier elements gives rise to the following formula for $N_j(x)$.

Lemma I.3: Let I_j denote the integer part and f_j the fractional part of $\frac{w_j B}{L}$. (In symbols $I_j = \left\lfloor \frac{w_j B}{L} \right\rfloor$ and $f_j = \frac{w_j B}{L} - \left\lfloor \frac{w_j B}{L} \right\rfloor$.) Then, *ignoring edge effects*, $N_j(x)$ equals I_j on a set of measure $(1-f_j)L$, and equals $I_j + 1$ on a set of measure $f_j L$. Alternatively, if the random variable $N_j(\tilde{x})$ is the number of barrier elements that can potentially detect penetrator ρ , where \tilde{x} is the uniformly distributed crossing point of penetrator ρ , then

$$N_j(\tilde{x}) = \begin{cases} I_j & \text{w.p. } 1-f_j \\ I_j + 1 & \text{w.p. } f_j \end{cases}$$

Proof: See Appendix A. Section F, below, discusses the effect of edge effects on $N_j(x)$.

Combining the above results, we obtain

$$P(S) = (1-f_j)(1-P(K))^{I_j} + f_j(1-P(K))^{I_j+1},$$

which simplifies to

$$P(S) = (1-P(K))^{I_j} (1-f_j P(K)).$$

We note that $P(K)$ is itself a function of the type j of penetrator ρ . The expected number of penetrators of type j killed is

$$\begin{aligned} E(\tilde{R}_j) &= R_j (1-P(S)) \\ &= R_j [1-(1-P(K))^{I_j} (1-f_j P(K))]. \end{aligned}$$

The computation of $E(\tilde{y})$ proceeds in a similar manner. We again consider a specific penetrator ρ , of type j . Conditioning on the crossing point \tilde{x} of ρ , we obtain

$$E(\tilde{y}) = \int_0^L E(\tilde{y} | C_x) \frac{1}{L} dx.$$

For each x ,

$$E(\tilde{y} | C_x) = \sum_{b=1}^B E(\tilde{y}_b | C_x).$$

If x is such that barrier element b cannot potentially detect p , then b with certainty fires zero shots at p . If b can potentially detect p , then by the results of Subsection 1, the expected number of shots b fires at p is the quantity denoted by $E(\tilde{h})$. Therefore

$$E(\tilde{y} | C_x) = N_j(x) E(\tilde{h}).$$

This is true whether or not the independence condition holds. Integrating over x and recalling the formula for $N_j(x)$, we obtain

$$E(\tilde{y}) = (1-f_j) I_j E(\tilde{h}) + f_j (I_j + 1) E(\tilde{h}).$$

Noting that $I_j + f_j = \frac{w_j B}{L}$, we can simplify this to

$$E(\tilde{y}) = \frac{w_j B}{L} E(\tilde{h}).$$

This formula makes sense in that the average number of barrier elements that can potentially detect p is the average value of $N_j(\bar{x})$, or

$$(1-f_j) I_j + f_j (I_j + 1) = I_j + f_j = \frac{w_j B}{L}.$$

and each barrier element that can potentially detect p fires an average of $E(\tilde{h})$ shots at p .

To compute the probability distribution of \tilde{y} , we must assume that the premises of the independence condition hold, i.e., attack protocol (a) is used or $w_j \leq L/B$. If protocol (a) is used, each barrier element fires exactly one shot at each penetrator it detects. We have made the tacit assumption that detections of a given penetrator by different barrier elements are conditionally independent given the crossing point x of that penetrator. (They are clearly not independent overall.) Under protocol (a), then, if penetrator p crosses at x , the number of shots fired at p is binomially distributed with parameters $N_j(x)$ and d_j . That is

$$P(\tilde{y}=y | C_x) = \binom{N_j(x)}{y} d_j^y (1-d_j)^{N_j(x)-y},$$

for $y = 0, 1, \dots, N_j(x)$. Integrating over x , we obtain, after extensive algebra

$$P(\tilde{y}=y) = \begin{cases} \binom{I_j}{y} d_j^y (1-d_j)^{I_j-y} \left[1-f_j + \frac{f_j (1-d_j) (I_j+1)}{(I_j+1-y)} \right] & y=0, \dots, I_j \\ f_j d_j^{I_j+1} & y=I_j+1 \\ 0 & \text{for all other } y. \end{cases}$$

If $w_j \leq L/B$, then by Lemma I.3, $N_j(x)$ is either 0 or 1, and penetrator p is never vulnerable to more than one barrier element. If $N_j(x) = 0$, then no barrier element can potentially detect p , and \tilde{y} is zero with certainty. If $N_j(x) = 1$, then one barrier element, call it β , can potentially detect p , and \tilde{y} has the same distribution as \tilde{h} , which was derived in Section E.1. If $w_j < L/B$, $f_j = w_j B/L$, and $N_j(\tilde{x})$ is zero with probability $1 - f_j$. Thus

$$\begin{aligned} P(\tilde{y}=0) &= (1-f_j) \cdot 1 + f_j P(F_0) \\ &= 1 - \frac{w_j B}{L} (1-P(F_0)) \end{aligned}$$

and

$$\begin{aligned} P(\tilde{y}=y) &= (1-f_j) \cdot 0 + f_j P(F_y) \\ &= \frac{w_j B}{L} P(F_y), \quad y=1, \dots, H. \end{aligned}$$

It can easily be verified that these formulas are also valid when $w_j = L/B$, i.e., $I_j = 1$ and $f_j = 0$.

F. EDGE EFFECTS

Edge effects can affect the accuracy of the formulas for the three quantities of interest (the expected number of penetrators killed, the expected number of shots fired at a penetrator, and the probability distribution of shots fired at a penetrator) if some barrier elements are located less than $w_j/2$ away from the end of the barrier, for some penetrator types j . In this case, the "detectability width"--the interval of length w_j centered on the location of the barrier element--overlaps the area beyond the end of the barrier, where no

penetrators cross. In terms of the symbols defined in the preceding section, this has two major consequences:

1. The true value of $N_j(x)$ --the number of barrier elements that can potentially detect a penetrator of type j crossing at point x --is lower than the formula given previously. See the proof of Lemma I.3 in Appendix A for more discussion of $N_j(x)$.
2. For barrier elements b near (i.e., less than $w_j/2$ away from) the edge of the barrier, the probability that a given penetrator is vulnerable to barrier element b is not w_j/L but equals $1/L$ times ($w_j/2$ plus the distance of b from the barrier edge), a smaller quantity. This changes the probabilities $P(A_1)$ and $P(DA_1)$ used in computing $P(K)$, $E(\tilde{h})$, and $P(\tilde{h}=h)$.

It is not immediately obvious how these two effects affect the accuracy of the formulas for the overall quantities $E(\tilde{\Delta R}_j)$, $E(\tilde{y})$, and $P(\tilde{y}=y)$. However, the problem of edge effects may not be exceedingly important. If barrier elements are repositioned so that no element's detectability area extends beyond the ends of the barrier, the values of both $N_j(x)$ and $P(A_1)$ will increase, and the errors in these two quantities might compensate for each other. And if the independence condition holds, edge effects are often automatically eliminated, as described below.

By Assumption 1), if there are B barrier elements, the barrier line is divided into B equal segments and a barrier element is positioned at the midpoint of each segment. For a barrier of length L , if one endpoint is labeled with the coordinate value 0, then the position of barrier element b is

$$x_b = \frac{L}{B}(b - \frac{1}{2})$$

for $b=1, \dots, B$.

In particular, barrier elements 1 and B are positioned at $L/2B$ and $L(1-1/2B)$, respectively. Recall from Section D that the events S_b must be conditionally independent given the crossing point \tilde{x} of a penetrator if the formulas for $E(\tilde{\Delta R}_j)$ and $P(\tilde{y}=y)$ are to be valid. (Here, "valid" means that a formula is correct, possibly subject to Poisson approximation or edge effects.) This independence condition will hold if either

- (a) Attack protocol (a) is used (a barrier element fires exactly one shot at each penetrator it detects)

or

- (b) $w_j \leq L/B$ for the particular penetrator type j considered,

or both. As stated earlier, frequently w_j will be less than or equal to L/B for *all* j , specifically in the case where each barrier element is not allowed to attack penetrators outside the barrier element's "cell" of length L/b . Therefore the formulas for $E(\Delta\tilde{R}_j)$, $E(\tilde{y})$ and $P(\tilde{y}=y)$ are valid for all j . But also in this case, no barrier element is less than $w_j/2$ away from the barrier edge, for any j , and thus edge effects are not a problem either.

The issue of what happens when $w_j \leq L/B$ for some but not all j is more complicated. Consider penetrators of type j where $w_j \leq L/B$. The independence condition does hold, and, except for edge effects, the formulas for $E(\Delta\tilde{R}_j)$, $E(\tilde{y})$ and $P(\tilde{y}=y)$ are valid. We conjecture, however, that they are not subject to edge effects if and only if attack protocol (a) is used. The problem is that (in the derivation presented in this paper) for other attack protocols we must condition on the number of other penetrators a barrier element detects--and the appropriate probabilities of detection will be subject to edge effects.

Less holds true for penetrators of those types j for which $w_j > L/B$. If attack protocol (a) is used the independence condition holds, and the formulas for $E(\Delta\tilde{R}_j)$, $E(\tilde{y})$, and $P(\tilde{y}=y)$ are valid but subject to edge effects. If an attack protocol other than (a) is used, the formulas for $E(\Delta\tilde{R}_j)$ and $P(\tilde{y}=y)$ are not valid, and the formula for $E(\tilde{y})$ is subject to edge effects.

Edge effects will become more and more of a problem the greater the ratios $w_j/(L/B)$ are. For the distance of barrier element b from the "left-hand" barrier edge (point 0) is

$$\frac{L}{B}(b - \frac{1}{2})$$

and this will be less than $w_j/2$ away from this edge if

$$2b < 1 + \frac{w_j}{L/B}.$$

Thus the greater the ratio $w_j/(L/B)$, the more barrier elements b satisfy this inequality and thus have their detectability widths overlap the area beyond the left-hand barrier edge. Similar arguments apply to the right-hand barrier edge (point L).

G. THE RESULTS

This section presents the formulas for the three main quantities of interest--the expected number of penetrators killed, the expected number of shots fired at a given penetrator, and the probability distribution of the number of shots fired at a given penetrator. Results are given for the five attack protocols in both homogeneous ($n=1$) and heterogeneous ($n>1$) cases. Results for which we have not been able to find a closed form are expressed in terms of cumulative binomial or Poisson probabilities when possible. The methods of derivation of these formulas have been presented in Section E, above; proof details appear in Appendix A.

Note that if H is set equal to 1 in the formulas for attack protocols (b), (c), or (d) (and the appropriate algebraic simplifications are performed), the corresponding formula for attack protocol (e) results. Also, if H is extremely large, the formulas for protocol (d) reduce to the corresponding formulas for protocol (a). The descriptions of the attack protocols imply that the above results should hold; that they do is an independent check on the accuracy of the formulas.

1. Notation

This section briefly reviews some of the notation used in the formulas. The quantities for which formulas are presented are as follows:

$E(\Delta R_j)$ --expected number of penetrators of type j killed. (In the homogeneous case, simply $E(\Delta R)$.)

$E(\tilde{y})$ --expected number of shots fired at a specific penetrator. (In the heterogeneous case, the penetrator is assumed to be of some specific type j .)

$P(\tilde{y}=y)$ --the probability that exactly y shots are fired at a specific penetrator (of type j).

The following parameters are defined in the assumptions in Section A: L , w_j , d_j , H , k_j . In the homogeneous case, w , d , and k are used instead of w_j , d_j , and k_j , respectively. There are n types of penetrators, R_j penetrators of type j , R penetrators in total ($R = \sum R_j$), and B barrier elements.

The probabilities that a given barrier element detects a given detectable penetrator and exactly t other penetrators (the $P(DA_t)$) appear in some of the expressions that do not

have closed forms. For conciseness, we will use the notation ω_l and ψ_{jl} for them. In the homogeneous case we set

$$\omega_l = d \binom{R-1}{l} \left(\frac{wd}{L} \right)^l \left(1 - \frac{wd}{L} \right)^{R-1-l} \quad l=0, \dots, R-1.$$

We will often denote the quantity (wd/L) by p . In the heterogeneous case we first define, for each $j=1, \dots, n$,

$$\alpha_j = (R_j - 1) \frac{d_j w_j}{L} + \sum_{\substack{i=1 \\ i \neq j}}^n R_i \frac{d_i w_i}{L},$$

and then set

$$\psi_{jl} = d_j \frac{e^{-\alpha_j} \alpha_j^l}{l!}, \quad l=0, 1, 2, \dots$$

The following notation will be used for binomial and Poisson probabilities. (In this paragraph only, n is any nonnegative integer and p is any real number between zero and one.) We define

$$b(m; n, p) = \binom{n}{m} p^m (1-p)^{n-m}, \quad m=0, \dots, n$$

$$B(m; n, p) = \begin{cases} 0 & \text{for } m \leq -1 \\ \sum_{l=0}^m b(l; n, p) & \text{for } m=0, \dots, n \\ 1 & \text{for } m \geq n \end{cases}$$

$$p(m; \mu) = e^{-\mu} \mu^m / m!$$

and

$$P(m; \mu) = \begin{cases} 0 & \text{for } m \leq -1 \\ \sum_{l=0}^m p(l; \mu) & \text{for } m=0, 1, 2, \dots \end{cases}$$

Finally we will use the quantities I_j and f_j defined (in the heterogeneous case) for $j=1,\dots,n$ by

$$I_j = \left\lfloor \frac{w_j B}{L} \right\rfloor$$

$$f_j = \frac{w_j B}{L} - I_j$$

In the homogeneous case we use the analogous quantities I and f .

2. The Expected Number of Penetrators Killed

a. Homogeneous Case

Theorem I.1a. Suppose that $n=1$ and attack protocol (a) is used. Then, ignoring edge effects,

$$E(\tilde{\Delta R}) = R \left[1 - (1 - kd)^I (1 - fkd) \right]$$

Theorems I.1b, c, d, and e. Suppose that $n=1$ and $w \leq L/B$. $E(\tilde{\Delta R})$ is given by

$$E(\tilde{\Delta R}) = R_j \left[1 - (1 - P(K))^I (1 - fP(K)) \right],$$

where $P(K)$ is defined for the specific attack protocol used, as indicated below. (Note that $I=0$ or 1 , and if $I=1$, $f=0$.)

- Attack protocol (b)

$$P(K) = d - \sum_{i=0}^{R-1} \omega_i \left(\frac{1+1-k}{1+1} \right)^H$$

- Attack protocol (c) Define $U = \min(H-1, R-1)$, and let

$$A = \sum_{i=0}^U \omega_i (1-k) \left\lfloor \frac{H}{i+1} \right\rfloor (1 - \langle \frac{H}{i+1} \rangle k) .$$

Then if $H \geq R$, $P(K) = d - A$, and if $H \leq R-1$

$$P(K) = \frac{HkL}{wR} - A + dE(H-1; R-1, p) - \frac{HkL}{wR}B(H; R, p),$$

where $p = \frac{wd}{L}$.

- Attack protocol (d) Let $p = wd/L$. Then if $H \geq R$, $P(K) = kd$, and if $H \leq R-1$,

$$P(K) = kdE(H-1; R-1, p) + \frac{kHL}{wR}(1 - B(H; R, p)).$$

- Attack protocol (e)

$$P(K) = \frac{kL}{wR} \left[1 - \left(1 - \frac{dw}{L} \right)^R \right].$$

b. Heterogeneous Case

Theorem I.2a. Suppose that $n > 1$ and that attack protocol (a) is used. Then, ignoring edge effects,

$$E(\Delta R_j) = R_j [1 - (1 - k_j d_j)^{I_j} (1 - f_j k_j d_j)].$$

Theorems I.2b, c, d, and e. Suppose that $n > 1$ and $w_i \leq L/B$ for all $i = 1, \dots, n$. Then $E(\Delta R_j)$ for $j = 1, \dots, n$, is approximately given by

$$E(\Delta R_j) \approx R_j [1 - (1 - P(K))^{I_j} (1 - f_j P(K))]$$

where $P(K)$ is defined for the specific attack protocol used, as indicated below. (Note that

$I_j = 0$ or 1 , and if $I_j = 1$, $f_j = 0$.)

Attack protocol (b)

$$P(K) \approx d_j - \sum_{t=0}^{\infty} \left(1 - \frac{k_j}{t+1} \right)^H \psi_{jt}$$

Attack protocol (c) Define

$$A = \sum_{t=0}^{H-1} \psi_{jt} (1 - k_j)^{\left\lfloor \frac{H}{t+1} \right\rfloor} \left(1 - \left\langle \frac{H}{t+1} \right\rangle k_j \right).$$

Then if $\alpha_j > 0$

$$P(K) \approx \frac{Hk_j d_j}{\alpha_j} - A + d_j P(H-1; \alpha_j) - \frac{Hk_j d_j}{\alpha_j} P(H; \alpha_j)$$

and if $\alpha_j = 0$

$$P(K) \approx d_j [1 - (1 - k_j)^H].$$

(It is assumed that H is a positive integer.)

- Attack protocol (d)

$$P(K) \approx \begin{cases} k_j d_j P(H-1; \alpha_j) + \frac{k_j d_j H}{\alpha_j} (1 - P(H; \alpha_j)) & \text{if } \alpha_j > 0 \\ k_j d_j & \text{if } \alpha_j = 0 \end{cases}$$

- Attack protocol (e)

$$P(K) \approx \begin{cases} \frac{k_j d_j}{\alpha_j} (1 - e^{-\alpha_j}) & \text{if } \alpha_j > 0 \\ k_j d_j & \text{if } \alpha_j = 0. \end{cases}$$

3. The Expected Number of Shots Fired at a Given Penetrator

In this subsection, we consider a specific penetrator, which we call penetrator ρ . In the heterogeneous case, j denotes ρ 's type. The random variable \tilde{y} is the expected number of shots fired at penetrator ρ (summed over all barrier elements). Here we present formulas for $E(\tilde{y})$; in subsection 4, we present formulas for $P(\tilde{y}=y)$, for appropriate y .

a. Homogeneous Case

Theorems 1.3a, b, c, d, and e. Suppose that $n=1$. Then, ignoring edge effects, if attack protocol (a), (b), (c), (d), or (e) is used, as indicated below, $E(\tilde{y})$ is given by the following formulas:

- Attack protocol (a)

$$E(\tilde{y}) = \frac{wB}{L}d = B\frac{wd}{L}$$

- Attack protocol (b)

$$E(\tilde{y}) = \frac{BH}{R} [1 - (1 - \frac{wd}{L})^R]$$

- Attack protocol (c)

$$E(\tilde{y}) = \frac{BH}{R} [1 - (1 - \frac{wd}{L})^R]$$

- Attack protocol (d)

$$\text{If } H \geq R, E(\tilde{y}) = \frac{wBd}{L}, \text{ as in (a).}$$

$$\text{If } H \leq R-1, E(\tilde{y}) = \frac{wB}{L} [d\mathbb{B}(H-1; R-1, p) + \frac{HL}{wR} (1 - \mathbb{B}(H; R, p))],$$

where $p = wd/L$

- Attack protocol (e)

$$E(\tilde{y}) = \frac{B}{R} [1 - (1 - \frac{wd}{L})^R]$$

Note that attack protocols (b) and (c) have the same formula. (See Appendix A for more on this.) If $H=1$, it can be verified that the formula in (d) (and of course those in (b) and (c)) reduces to that in (e).

b. Heterogeneous Case

Theorem I.4a. Suppose that $n > 1$ and attack protocol (a) is used. Then, ignoring edge effects,

$$E(\tilde{y}) = \frac{w_j B d_j}{L}.$$

Theorems I.4b, c, d, and e. Suppose that $n > 1$. Then, ignoring edge effects, if attack protocol (b), (c), (d), or (e) is used, as indicated below, $E(\tilde{y})$ is approximately given by the following formulas.

- Attack protocol (b)

$$E(\tilde{y}) \approx \begin{cases} \frac{w_j B H d_j}{L \alpha_j} (1 - e^{-\alpha_j}) & \text{if } \alpha_j > 0 \\ w_j B H d_j / L & \text{if } \alpha_j = 0. \end{cases}$$

- Attack protocol (c)

$$E(\tilde{y}) \approx \begin{cases} \frac{w_j B H d_j}{L \alpha_j} (1 - e^{-\alpha_j}) & \text{if } \alpha_j > 0 \\ w_j B H d_j / L & \text{if } \alpha_j = 0. \end{cases}$$

as in (b).

- Attack protocol (d)

$$E(\tilde{y}) \approx \begin{cases} \frac{w_j B}{L} [d_j \mathbb{P}(H-1; \alpha_j) + \frac{H d_j}{\alpha_j} (1 - \mathbb{P}(H; \alpha_j))] & \text{if } \alpha_j > 0 \\ w_j B d_j / L & \text{if } \alpha_j = 0. \end{cases}$$

- Attack protocol (e)

$$E(\tilde{y}) \approx \begin{cases} \frac{w_j B d_j}{L \alpha_j} (1 - e^{-\alpha_j}) & \text{if } \alpha_j > 0 \\ w_j B d_j / L & \text{if } \alpha_j = 0. \end{cases}$$

4. The Probability Distribution of the Number of Shots Fired at a Given Penetrator

a. Attack Protocol (a)

Theorem I.6a. Suppose that $n > 1$ and attack protocol (a) is used. Then, ignoring edge effects and assuming that the given penetrator is of type j ,

$$P(\tilde{y}=y) = \begin{cases} \binom{I_j}{y} d_j^y (1-d_j)^{I_j-y} \left[1 - f_j + \frac{f_j (1-d_j) (I_j+1)}{I_j+1-y} \right], & y=0, \dots, I_j \\ f_j d_j^{I_j+1}, & y=I_j+1. \end{cases}$$

Theorem I.5a. Suppose that $n=1$ and attack protocol (a) is used. Then, subject to edge effects, the formulas for the probabilities $P(\tilde{y}=y)$ are as given in Theorem I.6a, but substituting d , I , and f for d_j , I_j , and f_j , respectively.

b. Attack Protocols (b), (c), (d), (e)

Before presenting the results for $P(\tilde{y}=y)$ for these cases, we relate the $P(\tilde{y}=y)$ to the probabilities $P(F_h)$, where F_h is the event that a specific penetrator (penetrator ρ) receives exactly h shots from a specific barrier element β that can potentially detect ρ . The random variables

\tilde{h}_b = number of shots fired at penetrator ρ by barrier element b

for $b=1, \dots, B$ are in general not independent, and not even conditionally independent given the crossing point x of ρ . To avoid problems with this, we assume that $w_i \leq L/B$ for all penetrator types $i=1, \dots, n$, so that penetrator ρ is detectable by at most one barrier element. The following lemma, proved in Appendix A, is valid in both the homogeneous and heterogeneous cases; in the former case, merely suppress the subscripts i and j . As before, j denotes the type of the specific penetrator ρ under consideration. For protocols (a) and (e), consider H as 1.

Lemma L4: Suppose $w_i \leq L/B$ for all i . Then, for all attack protocols,

-If $w_j < L/B$

$$P(\tilde{y}=0) = 1 - f_j + f_j P(F_0)$$

$$P(\tilde{y}=y) = f_j P(F_y), \quad y=1, \dots, H$$

- If $w_j = L/B$

$$P(\tilde{y}=y) = P(F_y) \quad y=0, \dots, H.$$

Recall that I_j and f_j are defined as the integer and fractional parts, respectively, of $w_j B/L$. If $w_j < L/B$, then $I_j=0$ and $f_j=w_j B/L$. If $w_j=L/B$, then formally $I_j=1$ and $f_j=0$, but letting $I_j=0$ and $f_j=1$ in the first set of formulas yields the second set of formulas. Accordingly, for the remainder of this section only, we redefine, for $j=1, \dots, n$,

$$f_j = \begin{cases} \frac{w_j B}{L} & \text{if } w_j < L/B \\ 1 & \text{if } w_j = L/B. \end{cases}$$

In the homogeneous case, we define f by the above formulas, suppressing the subscript j .

Utilizing derived expressions for the $P(F_y)$, we can state the following theorems for the homogeneous and heterogeneous cases, respectively.

Theorems 1.5b, c, d, and e. Suppose $n=1$ and $w \leq L/B$. Then the random variable \tilde{y} has the probability distribution

$$P(\tilde{y}=0) = 1 - f + fP(F_0)$$

$$P(\tilde{y}=y) = fP(F_y), \quad y=1, \dots, H,$$

where the $P(F_y)$ are defined separately for attack protocols (b), (c), (d), and (e), as indicated below:

- Attack protocol (b).

$$P(F_0) = 1-d + \sum_{i=0}^{R-1} \omega_i \left(\frac{1}{i+1} \right)^H$$

$$P(F_y) = \sum_{i=0}^{R-1} \omega_i \binom{H}{y} \left(\frac{1}{i+1} \right)^y \left(\frac{1}{i+1} \right)^{H-y} \quad y=1, \dots, H.$$

- Attack protocol (c). Let $[x]^+$ denote the positive part of x , i.e.,

$$[x]^+ = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and define $p=wd/L$. Then

$$P(F_0) = 1-d + d(1 - \mathbb{B}(H-1; R-1, p)) - \frac{HL}{wR} (1 - \mathbb{B}(H; R, p)).$$

(Note that if $H \geq R$, this formula becomes $1-d$, which is appropriate considering the meaning of H , R , and F_0 .)

$$\begin{aligned} P(F_1) &= 2d(\mathbb{B}(H-1; R-1, p) - \mathbb{B}(\lfloor \frac{H}{2} \rfloor - 1; R-1, p)) \\ &\quad - \frac{HL}{wR} (\mathbb{B}(H; R, p) - \mathbb{B}(\lfloor \frac{H}{2} \rfloor; R, p)) \\ &\quad + \frac{HL}{wR} (1 - \mathbb{B}(H; R, p)) \end{aligned}$$

and if $H \geq 2$, then for $y=2, \dots, H$

$$\begin{aligned} P(F_y) = & (y+1) d \left[\mathbb{E}\left(\left\lfloor \frac{H}{y} \right\rfloor - 1; R-1, p\right) - \mathbb{E}\left(\left\lfloor \frac{H}{y+1} \right\rfloor - 1; R-1, p\right) \right]^+ \\ & - \frac{HL}{wR} \left[\mathbb{E}\left(\left\lfloor \frac{H}{y} \right\rfloor; R, p\right) - \mathbb{E}\left(\left\lfloor \frac{H}{y+1} \right\rfloor; R, p\right) \right]^+ \\ & + \frac{HL}{wR} \left[\mathbb{E}\left(\left\lfloor \frac{H}{y-1} \right\rfloor; R, p\right) - \mathbb{E}\left(\left\lfloor \frac{H}{y} \right\rfloor; R, p\right) \right]^+ \\ & - (y-1) d \left[\mathbb{E}\left(\left\lfloor \frac{H}{y-1} \right\rfloor - 1; R-1, p\right) - \mathbb{E}\left(\left\lfloor \frac{H}{y} \right\rfloor - 1; R-1, p\right) \right]^+. \end{aligned}$$

- Attack protocol (d). Let $p=wd/L$. Then

$$P(F_0) = 1 - P(F_1)$$

where

$$\begin{aligned} P(F_1) = & d \mathbb{E}(H-1; R-1, p) + \frac{HL}{wR} (1 - \mathbb{E}(H; R, p)). \\ P(F_y) = & 0, y \geq 2. \end{aligned}$$

Note that if $H \geq R$, $P(F_1)$ becomes d , which yields the same result as protocol (a) and is consistent with the definitions of H , R , and F_1 .

- Attack protocol (e). Again, let $p=wd/L$. Then

$$P(F_0) = 1 - P(F_1),$$

where

$$\begin{aligned} P(F_1) = & \frac{L}{wR} (1 - (1-p)^R). \\ P(F_y) = & 0, y \geq 2. \end{aligned}$$

It can be verified that if $H=1$, the formulas for $P(F_0)$ and $P(F_1)$ in attack protocols (b), (c), and (d) reduce to those in protocol (e).

Theorems I.6b, c, d, and e. Suppose that $n > 1$ and $w_i \leq L/B$ for $i=1, \dots, n$.

If the specific penetrator ρ under consideration is of type j , then the random variable \bar{y} has the distribution

$$P(\bar{y}=0) = 1 - f_j + f_j P(F_0)$$

$$P(\bar{y}=y) = f_j P(F_y), \quad y=1, \dots, H,$$

where the $P(F_y)$ are approximately given by the expressions below, defined separately for attack protocols (b), (c), (d), and (e).

- Attack protocol (b)

$$P(F_0) \approx 1 - d_j + \sum_{i=0}^{\infty} \left(\frac{1}{i+1}\right)^H \psi_j$$

$$P(F_y) \approx \sum_{i=0}^{\infty} \binom{H}{y} \left(\frac{1}{i+1}\right)^y \left(\frac{1}{i+1}\right)^{H-y} \psi_j, \quad y=1, \dots, H.$$

- Attack protocol (c). Let $[x]^+$ denote the positive part of x , as explained in Theorem I.5c. Then if $\alpha_j > 0$

$$\begin{aligned} P(F_0) &\approx 1 - d_j + d_j [1 - \mathbb{P}(H-1; \alpha_j)] - \frac{H d_j}{\alpha_j} [1 - \mathbb{P}(H; \alpha_j)] \\ P(F_1) &\approx 2 d_j [\mathbb{P}(H-1; \alpha_j) - \mathbb{P}(\lfloor \frac{H}{2} \rfloor - 1; \alpha_j)] \\ &\quad - \frac{H d_j}{\alpha_j} [\mathbb{P}(H; \alpha_j) - \mathbb{P}(\lfloor \frac{H}{2} \rfloor; \alpha_j)] \\ &\quad + \frac{H d_j}{\alpha_j} [1 - \mathbb{P}(H; \alpha_j)] \end{aligned}$$

and if $H \geq 2$, then for $y=2, \dots, H$

$$\begin{aligned} P(F_y) \approx & (y+1)d_j \left[\mathbb{P}\left(\left\lfloor \frac{H}{y} \right\rfloor - 1; \alpha_j\right) - \mathbb{P}\left(\left\lfloor \frac{H}{y+1} \right\rfloor - 1; \alpha_j\right) \right]^+ \\ & - \frac{Hd_j}{\alpha_j} \left[\mathbb{P}\left(\left\lfloor \frac{H}{y} \right\rfloor; \alpha_j\right) - \mathbb{P}\left(\left\lfloor \frac{H}{y+1} \right\rfloor; \alpha_j\right) \right]^+ \\ & + \frac{Hd_j}{\alpha_j} \left[\mathbb{P}\left(\left\lfloor \frac{H}{y-1} \right\rfloor; \alpha_j\right) - \mathbb{P}\left(\left\lfloor \frac{H}{y} \right\rfloor; \alpha_j\right) \right]^+ \\ & - (y-1)d_j \left[\mathbb{P}\left(\left\lfloor \frac{H}{y-1} \right\rfloor - 1; \alpha_j\right) - \mathbb{P}\left(\left\lfloor \frac{H}{y} \right\rfloor - 1; \alpha_j\right) \right]^+ \end{aligned}$$

If $\alpha_j=0$, then $P(F_0)=1-d_j$, $P(F_H)=d_j$, and $P(F_y)=0$ for all other values of y .

- Attack protocol (d)

$$P(F_0) = 1 - P(F_1),$$

where

$$P(F_1) \approx \begin{cases} d_j \mathbb{P}(H-1; \alpha_j) + \frac{Hd_j}{\alpha_j} (1 - \mathbb{P}(H; \alpha_j)) & \text{if } \alpha_j > 0 \\ d_j & \text{if } \alpha_j = 0. \end{cases}$$

$$P(F_y) = 0, \quad y \geq 2.$$

- Attack protocol (e)

$$P(F_0) \approx \begin{cases} 1 - \frac{d_j}{a_j} (1 - e^{-\alpha_j}) & \text{if } \alpha_j > 0 \\ 1 - d_j & \text{if } \alpha_j = 0 \end{cases}$$

$$P(F_1) \approx \begin{cases} \frac{d_j}{\alpha_j} (1 - e^{-\alpha_j}) & \text{if } \alpha_j > 0 \\ d_j & \text{if } \alpha_j = 0 \end{cases}$$

$$P(F_y) = 0, \quad y \geq 2.$$

II. BARRIER ELEMENTS ARE TARGETS, PENETRATORS ARE ATTACKERS

We now examine the attrition to barrier elements during the barrier crossing process. The notation is similar (and sometimes identical) to that of Chapter I. We still speak of barrier elements and penetrators. In this chapter, however, the terms "barrier element" and "target" will be used synonymously, and the terms "penetrator", "attacker", and "shooter" will be used synonymously. The statement of the problem in Section A, below, is almost identical to the corresponding statement in Chapter I, but the penetrators and barrier elements have reversed shooter and target roles. Some of the implications of this are discussed in Section B, below.

A. THE PROBLEM

Suppose that there are n types of penetrators (n could equal 1) with R_j penetrators of type j . Let

$$R = \sum_{j=1}^n R_j$$

denote the total number of penetrators. There is one type of barrier element; there are B barrier elements present. We consider five attrition processes, indexed by attack protocols (a) through (e) in Assumption 5) below, that proceed according to the following assumptions.

- 1) Barrier elements are spaced evenly along a barrier of length L , i.e., the line of length L is divided into B equal segments and a barrier element is positioned at the center of each segment.
- 2) At a given time, each penetrator picks a crossing point x somewhere along the barrier line, according to a uniform distribution, and all penetrators attempt to cross the barrier simultaneously.
- 3) The crossing points of different penetrators are mutually independent random variables.
- 4) If the crossing point of a given penetrator of type j lies within distance $w_j/2$ of a given barrier element, that penetrator will detect the barrier element with probability d_j . Otherwise, that penetrator will not detect that barrier element. Each penetrator detects the barrier elements that lie within $w_j/2$ of the penetrator's crossing point independently of one another.

- 5) Of the barrier elements it has detected, each penetrator chooses certain ones to attack with certain numbers of shots (salvos), according to one of the attack protocols described below. In protocols (b), (c), and (d), the parameter H_j is the number of salvos available per penetrator of type j . (All penetrators use the same protocol.)
 - (a) A penetrator fires exactly one salvo at each barrier element it detects.
 - (b) Of the barrier elements it has detected, each penetrator chooses one according to a uniform distribution and fires one salvo at it. The penetrator performs this process exactly H_j times. The target choices for successive firings are independent.
 - (c) If a penetrator detects m targets (barrier elements) it fires $\left\lfloor \frac{H_j}{m} \right\rfloor$ salvos at each target, then chooses according to a uniform distribution $H_j - \left\lfloor \frac{H_j}{m} \right\rfloor m$ of the m targets and fires one additional salvo at each.
 - (d) If the number of targets, m , that a penetrator detects does not exceed H_j , the shooter fires exactly one salvo at each target. If $m > H_j$, the shooter picks (randomly and uniformly) H_j of the m targets and fires one salvo at each.
 - (e) Of the targets it has detected, a penetrator chooses one target according to a uniform distribution and fires exactly one salvo at it. (This is the special case of protocols (b), (c), and (d) when $H_j = 1$.)
- 6) Conditional on detection of that barrier element, the decisions of different penetrators to attack a given barrier element are independent.
- 7) A shot (salvo) fired by a penetrator of type j at a barrier element kills the barrier element with probability k_j . The effects of different shots (fired at the same barrier element) are independent.

We wish to compute the following quantities:

- The expected number of barrier elements killed.
- The expected number of shots (salvos) fired at a given barrier element.
- The probability distribution of the number of shots fired at a given barrier element.

Exact or approximate formulas for the first two of these quantities have been developed for each attack protocol. General formulas for the probability distribution of the number of shots fired at a given barrier element have been derived only for attack protocols (a), (d), and (e). As in Chapter I, we have sometimes ignored edge effects when deriving these formulas. Throughout this chapter and Appendix B, we indicate the derivation steps and resultant expressions where edge effects have been ignored. Section D discusses some of the consequences of ignoring edge effects.

Without loss of continuity, the reader can skip to Section E, which presents the formulas that have been developed for the three quantities of interest.

B. COMPARISON WITH CHAPTER I

The notation for the problem statement has been chosen to reflect the complementary nature of the combat processes examined in the two chapters. The parameters n , R_j , R , B , and L denote exactly the same quantities in Chapter II as in Chapter I, and the same restrictions noted in Chapter I, Section B apply to them. The parameters w_j , d_j , H_j , and k_j denote in Chapter II quantities similar to the corresponding ones of Chapter I, but they apply to penetrators rather than barrier elements.

In this paper, penetrators operate (mutually) independently of each other, but barrier elements are positioned in a fixed pattern (evenly spaced along a line). There can be several types of penetrators but just one type of barrier element. In the current chapter, the penetrators are the shooters, so different shooters operate independently. This makes it easier than in Chapter I to derive the expected number of targets killed and the expected number of shots fired at a given target. No special independence condition is necessary. Since there is only one target type there is no need to use the Poisson approximation to the binomial to obtain tractable formulas for the number of targets a shooter detects. (This approximation is used in computing the probability distribution of the number of shots fired at a target in the case of heterogeneous shooters.) It is relatively easy to deal with multiple shooter types. Even so, the similarity of many of the final formulas to the corresponding ones in Chapter I is striking.

We use the words "detectability," "vulnerability," et al., in a manner similar to Chapter I. A penetrator's "detectability width" or "detectability area" is the line of length w_j (j is the penetrator type) centered on the crossing point \tilde{x} of that penetrator (\tilde{x} is a random variable). Any barrier element that lies within the detectability width of a given penetrator (in a certain realization of the process) is said to be detectable by that penetrator or vulnerable to that penetrator, or we say that that penetrator can potentially detect such a barrier element. If penetrator p can potentially detect barrier element β , p actually detects β with probability d_j .

C. DERIVATION OF THE DESIRED QUANTITIES

This section presents the probabilistic arguments used to derive formulas for the three basic quantities of interest: the expected number of barrier elements killed, the

expected number of shots fired at a given barrier element, and (where possible) the probability distribution of the number of shots fired at a given barrier element. The algebraic evaluation of the final formulas is presented in Appendix B; the formulas are presented as theorems in Section E, below.

The derivation proceeds in two parts. Subsection 1 considers one specific target (barrier element β) and one specific shooter (penetrator ρ , assumed to be of some type j). Subsection 2 integrates the results of Subsection 1 into formulas for the overall quantities of interest.

1. One Penetrator, One Barrier Element

Consider one specific barrier element β , and one specific penetrator ρ . We assume that ρ is of type j . There is only one type of barrier element, and we are tacitly assuming that the results here do not depend on the specific barrier element considered. This is true in the absence of edge effects. For this section we ignore edge effects; some of the complications they can cause are discussed in Section D. Unlike Chapter I, we do not assume that β is detectable by ρ . Instead we explicitly consider the probability of vulnerability.

We define the following events:

V -- ρ can potentially detect β , i.e., ρ 's crossing point is within $w_j/2$ of the location of β .

W_m --The total number of barrier elements that ρ can potentially detect is exactly m . (Defined for $m=0, 1, \dots, B$.)

VW_m --The intersection of V and W_m , i.e., ρ can potentially detect β and $m-1$ other barrier elements.

D -- ρ actually detects β .

A_t -- ρ detects exactly t targets *other* than target β .

DA_t --The intersection of D and A_t , i.e., ρ detects target β and exactly t other targets, for a total of $t+1$. (Defined for $t=0, \dots, B-1$.)

F_h -- ρ fires exactly h shots at β . (Defined for $h=0, \dots, H_j$.)

K -- ρ kills β .

We also let the random variable \tilde{h} denote the number of shots ρ fires at β , so

$$P(F_h) = P(\tilde{h}=h).$$

The quantities we derive here are $P(K)$, $E(\tilde{h})$, and the $P(F_h)$ for $h=0, \dots, H_j$. These will be used as inputs to Subsection 2.

By Assumption 5), target β will receive no more than H_j shots from shooter ρ (in attack protocols (a) and (e), $H_j=1$ for all j) so by Assumption 7)

$$\begin{aligned} P(K) &= \sum_{h=0}^{H_j} (1 - [1 - k_j]^h) P(F_h) \\ &= \sum_{h=1}^{H_j} (1 - [1 - k_j]^h) P(F_h). \end{aligned}$$

Similarly

$$E(\tilde{h}) = \sum_{h=1}^{H_j} h P(F_h).$$

In attack protocols (b), (c), (d), and (e), the number of shots ρ fires at β depends on the total number of targets ρ has detected, or equivalently, the number ι of targets other than β that ρ has detected. Assumption 5) of Chapter II is identical to Assumption 5) of Chapter I, except that penetrators and barrier elements have reversed shooter and target roles. Thus the probabilities

$$P(F_h | DA_\iota)$$

for $h=0, \dots, H_j$ and the appropriate values of ι are identical to those used in Chapter I, for each attack protocol. These probabilities are presented in Lemma I.5 of Appendix A. (To use these in context here, substitute H_j for H .) Since \bar{D} and $\{DA_\iota\}_{\iota \geq 0}$ form a partition, and a shooter fires zero shots at a target it does not detect, by the law of total probability

$$P(F_0) = P(\bar{D}) + \sum_{\iota} P(F_0 | DA_\iota) P(DA_\iota)$$

and

$$P(F_h) = \sum_i P(F_h | DA_i) P(DA_i), \quad h=1, \dots, H.$$

To derive the probabilities $P(DA_i)$, we first determine the number of targets that ρ can potentially detect considering the spacing of the targets along the barrier line. Clearly, ignoring edge effects, $P(V) = w_j/L$ because ρ 's crossing point is uniformly distributed on $[0, L]$ (Assumptions 2) and 4)). We note that (by Assumption 4)) $P(D) = d_j P(V) = w_j d_j / L$. The probabilities of the W_m ($m=0, \dots, B$) can be computed by methods similar to those used in Chapter I, but V and W_m are, in general, not independent. The following lemma, which is of key importance in the derivation, is proved in Appendix B.

Lemma II.1 Let $I_j = \lfloor w_j B / L \rfloor$ and $f_j = \langle w_j B / L \rangle$. Then, *ignoring edge effects*,

$$P(W_{I_j} | V) = \frac{I_j (1 - f_j)}{(w_j B / L)}$$

$$P(W_{I_j+1} | V) = \frac{(I_j + 1) f_j}{(w_j B / L)}$$

$$P(W_m | V) = 0 \text{ for all other values of } m.$$

(Parameter L is restricted to be > 0 . If $w_j=0$, then penetrator ρ cannot potentially detect any target, and the joint probabilities $P(VW_m)$ are zero for all m .) We also state the probabilities of the W_m .

Lemma II.2 Let I_j and f_j be defined as in Lemma II.1. Then, *ignoring edge effects*,

$$P(W_{I_j}) = 1 - f_j$$

$$P(W_{I_j+1}) = f_j$$

$$P(W_m) = 0 \text{ for all other values of } m.$$

Lemma II.2 is proved in Appendix B. The problem of edge effects is discussed in Section D, below. The joint probabilities $P(VW_{I_j})$ and $P(VW_{I_j+1})$ are then

$$P(VW_{I_j}) = P(W_{I_j} | V) P(V) = \frac{I_j (1 - f_j)}{B}$$

and

$$P(VW_{I_j+1}) = P(W_{I_j+1}|V) P(V) = \frac{(I_j+1) f_j}{B}.$$

Suppose that VW_m occurs. Then the event DA_l means that p detects β and exactly l of the $m-1$ other targets vulnerable to p . By Assumption 4), detections of different vulnerable targets are independent and each occurs with probability d_j . Therefore the number of other targets detected follows a binomial distribution with parameters $m-1$ and d_j , and

$$P(DA_l|VW_m) = d_j \binom{m-1}{l} d_j^l (1-d_j)^{m-1-l}, \quad l=0, \dots, m-1.$$

Since the event VW_m has positive probability only if $m=I_j$ or I_j+1 , unconditionally,

$$P(DA_l) = P(DA_l|VW_{I_j}) P(VW_{I_j}) + P(DA_l|VW_{I_j+1}) P(VW_{I_j+1}).$$

Substituting the appropriate derived expressions and simplifying yields

$$P(DA_l) = \frac{I_j(1-f_j)}{B} \binom{I_j-1}{l} d_j^{l+1} (1-d_j)^{I_j-l-1} \\ + \frac{(I_j+1)f_j}{B} \binom{I_j}{l} d_j^{l+1} (1-d_j)^{I_j-l-1} \quad l=0, \dots, I_j-1$$

and

$$P(DA_{I_j}) = \frac{(I_j+1)f_j}{B} d_j^{I_j+1}.$$

If $I_j=0$, then the first formula above does not make sense. But in this case, at most one target can be vulnerable to penetrator p . The occurrence of D implies that target β is vulnerable to p , thus β is that one target and there are no other targets that p can potentially detect. Thus if $I_j=0$, DA_l is the null event for $l \geq 1$, and DA_0 is simply D , so $P(DA_0) = w_j d_j / L$.

Note also that if $I_j > 0$ and $d_j=1$, $P(DA_l)$ will be zero for all l except I_j-1 and I_j . This results in considerable simplification of several formulas.

Overall, for $h \geq 1$

$$P(F_h) = \sum_{i=0}^{I_j} P(F_h | DA_i) P(DA_i).$$

Substituting this expression into the expressions for $P(K)$ and $E(\tilde{h})$ we obtain

$$P(K) = \sum_{h=1}^{H_j} \sum_{i=0}^{I_j} (1 - [1 - k_j]^h) P(F_h | DA_i) P(DA_i)$$

and

$$E(\tilde{h}) = \sum_{h=1}^{H_j} \sum_{i=0}^{I_j} h P(F_h | DA_i) P(DA_i).$$

As in Chapter I, these expressions are most easily evaluated by interchanging the order of summation. The quantities

$$P(K | DA_i) = \sum_{h=1}^{H_j} (1 - [1 - k_j]^h) P(F_h | DA_i)$$

and

$$E(\tilde{h} | DA_i) = \sum_{h=1}^{H_j} h P(F_h | DA_i)$$

(the lower limits of summation could be zero as well as one) are exactly the same as the corresponding quantities of Chapter I.¹ Expressions for them (for each different attack protocol) appear in Table A-1 of Appendix A. Then

$$P(K) = \sum_{i=0}^{I_j} P(K | DA_i) P(DA_i)$$

and

$$E(\tilde{h}) = \sum_{i=0}^{I_j} E(\tilde{h} | DA_i) P(DA_i),$$

and these quantities are evaluated simply by substituting the appropriate previously derived expressions (noting the i where $P(DA_i) = 0$) and simplifying. Results for each protocol are presented in Appendix B.

¹ The formulas are the same algebraically. However, parameters w_j , k_j , and d_j now apply to penetrators, not barrier elements, as shooters, and will in general have different numerical values from the corresponding Chapter I parameters, so the resultant values will differ. Also, H_j should be substituted for H .

2. All Penetrators and Barrier Elements

To use the results of the previous section to find the overall quantities of interest, we introduce new symbols for $P(K)$, $E(\tilde{h})$, and $P(\tilde{h}=h)$, as follows. For $j=1, \dots, n$ define

q_j --probability a given barrier element is killed by a given penetrator of type j .
(Ignoring edge effects, this probability is the same for all barrier elements.)

e_j --expected number of shots a given penetrator of type j fires at a given barrier element.

ϕ_{hj} --probability a given penetrator of type j fires exactly h shots at a given barrier element. (Defined for $h=0, \dots, H_j$.)

Also define:

$\Delta \tilde{B}$ --Number of barrier elements killed

$E(\Delta \tilde{B})$ --Expected number of barrier elements killed

S --Event that a given barrier element survives every penetrator

\tilde{y} --Number of shots fired at a given barrier element, totaled over all penetrators.

We seek expressions for $E(\Delta \tilde{B})$, $E(\tilde{y})$, and $P(\tilde{y}=y)$ for appropriate values of y . We present the main issues of derivation for each quantity in turn.

By elementary arguments [12],

$$E(\Delta \tilde{B}) = BP(\bar{S}) = B(1-P(S)).$$

The event S is the intersection of the events that the given barrier element survives a given penetrator, taken over all penetrators. By Assumptions 3) and 6), these individual survival events are independent. Since there are R_j penetrators of type j , for $j=1, \dots, n$,

$$P(S) = \prod_{j=1}^n (1-q_j)^{R_j}.$$

Thus

$$E(\Delta \tilde{B}) = B \left[1 - \prod_{j=1}^n (1-q_j)^{R_j} \right].$$

Similarly, \tilde{y} is the total over all penetrators of the random variables \tilde{h}_ρ , the number of shots fired at the given barrier element by penetrator ρ ($\rho=1,\dots,R$). If ρ is of type j , $E(\tilde{h}_\rho) = e_j$, as defined above. Therefore

$$E(\tilde{y}) = \sum_{j=1}^n R_j e_j .$$

In fact, by Assumptions 3) and 6), the random variables \tilde{h}_ρ , for different ρ are mutually independent. In attack protocols (a), (d), and (e), each \tilde{h}_ρ can assume only the values zero and one, and if ρ is of type j

$$P(\tilde{h}_\rho = 1) = \phi_{1j} ,$$

and the ϕ_{1j} can be different for different j . If there is only one type of penetrator, $\tilde{y} = \sum \tilde{h}_\rho$ follows a binomial distribution with parameters R (i.e., R_1) and ϕ_{11} . If $n > 1$, we can use the Poisson approximation to the binomial and say that \tilde{y} is approximately distributed Poisson with mean

$$\mu = \sum_{j=1}^n R_j \phi_{1j} .$$

The above discussion applies only if attack protocol (a), (d), or (e) is used. If attack protocol (b) or (c) is used, each \tilde{h}_ρ can assume a range of values, and even though the \tilde{h}_ρ are independent for different ρ , we have not obtained the resulting distribution for \tilde{y} . One simplifying case occurs if $w_j \leq L/B$ for all penetrator types j . In this case, each penetrator can potentially detect no more than one barrier element, and thus will fire all of its H_j shots at any barrier element it detects. Thus if ρ is of type j , \tilde{h}_ρ can assume only the values 0 or H_j . If in addition there is only one type of penetrator, then \tilde{y} , the number of shots fired at a given barrier element, equals H_1 multiplied by the number of penetrators that detect that barrier element. If $I_1=0$, this latter quantity is a binomial random variable with parameters R (i.e., R_1) and $\phi_{H_1 1}$. Therefore, if $n=1$, $I=0$, and attack protocol (b) or (c) is used,

$$P(\tilde{y}=mH_1) = \binom{R}{m} \phi_{H_1 1}^m (1-\phi_{H_1 1})^{R-m} . \quad m=0,\dots, R.$$

All of the above formulas were expressed in terms of the q_j , e_j , and ϕ_{hj} . The algebraic expressions derived in Subsection 1 for $P(K)$, $E(\tilde{h})$, and $P(\tilde{h}=h)$ can be

substituted in and overall algebraic formulas computed. The final results are presented in Section E; appropriate details appear in Appendix B.

D. EDGE EFFECTS

Edge effects will occur if some barrier elements are located less than $w_j/2$ away from the edge of the barrier, for some penetrator type j --i.e., if $w_j > L/B$. This condition is similar to the condition in Chapter I, Section F, but w_j is now a *penetrator's* detectability width. This has two consequences, which work in opposing directions. Consider a specific barrier element β located less than $w_j/2$ away from the barrier edge, for some j , and consider a penetrator p of type j . The probability $P(V)$ that p can potentially detect β is lower than w_j/L , which was used in the computations in Section C. However, the total number of barrier elements p can potentially detect assumes values lower than I_j or $I_j + 1$ if p 's "detectability width" extends beyond the barrier edge. If an attack protocol is used in which an attacker has a limited number of shots, if p does detect β there are fewer other detectees to divert p 's shots, and the probability β is killed increases.

If edge effects are considered, different penetrators of the same type can still be considered as identical (and mutually independent), but different barrier elements cannot. The probability q_j that a specific penetrator of type j kills a specific barrier element must be replaced by q_{jb} , the probability that the penetrator kills barrier element b , for $b=1, \dots, B$. Then

$$E(\bar{\Delta}B) = \sum_{b=1}^B [1 - \prod_{j=1}^n (1 - q_{jb})^{R_j}].$$

Similar arguments apply to the expectation and probability distribution of \tilde{y} .

If $w_j \leq L/B$, then edge effects do not occur for penetrators of type j , and the probabilities $P(VW_m)$ are as given in Section C. The reason for this is discussed in Appendix B. Since the overall quantities of interest involve sums or products on j , w_j must be less than or equal to L/B for all j if edge effects are not to occur. (Unlike Chapter I, this condition is not needed to guarantee the independence of any set of events.) The phrase "ignoring edge effects" in the statements of the theorems in the next section merely means that if $w_j \leq L/B$ for all j , the formulas are exact; if not, the formulas are valid except for edge effects.

We note finally that although the combat processes examined in this paper considered barrier elements spaced along a line, one could also consider a circular barrier or

screen, with barrier elements on the circumference (which is of length L). If detectability areas are considered as portions of the circumference, the results in both Chapters I and II hold without edge effects regardless of the relations between the w_j and L/B .

E. THE RESULTS

This section presents the formulas for the three main quantities of interest: $E(\tilde{\Delta}B)$, the expected number of barrier elements killed; $E(\tilde{y})$, the expected number of shots fired at a given barrier element; and the probabilities $P(\tilde{y}=y)$ that exactly y shots are fired at a given barrier element, for appropriate values y . (The $P(\tilde{y}=y)$ have been derived in general only for attack protocols (a), (d), and (e).) The methods of derivation of the formulas have been presented in Section C; proof details appear in Appendix B. Results for which we have not found a closed form are expressed with cumulative binomials, when possible.

The meanings of the parameters B , R , n , L , w_j , d_j , H_j , and k_j are given in Section A, above. We let I_j and f_j denote the integer and fractional parts, respectively, of $w_j B/L$. Note that $I_j = 0$ if, and only if, $w_j < L/B$. It can easily be verified, however, that the formulas given below for the case where $I_j = 0$ also hold if $w_j = L/B$, i.e., $I_j = 1$ and $f_j = 0$. For those penetrator types j where $I_j \geq 1$, we define the parameters

$$\begin{aligned} \theta_{j,t} = & \frac{1}{B} I_j (1-f_j) d_j \binom{I_j-1}{t} d_j^t (1-d_j)^{I_j-1-t} \\ & + \frac{1}{B} (I_j+1) f_j d_j \binom{I_j}{t} d_j^t (1-d_j)^{I_j-t} \quad \text{for } t=0, \dots, I_j-1, \end{aligned}$$

and

$$\theta_{j,I_j} = \frac{1}{B} (I_j+1) f_j d_j^{I_j+1}.$$

(Note that $\theta_{j,t}$ is the probability $P(\Delta A_t)$ used in Section C.) As in Chapter I, Section G, we use the following notation for binomial probabilities:

$$b(m; n, p) = \binom{n}{m} p^m (1-p)^{n-m} \quad m=0, \dots, n$$

and

$$B(m;n,p) = \begin{cases} 0, & m \leq -1 \\ \sum_{i=0}^m b(i; n,p), & m=0, \dots, n \\ 1, & m \geq n. \end{cases}$$

The theorems are indexed by attack protocol, reflecting dependence of the results on the attack protocol. The expected number of penetrators killed is given by the following theorems.

Theorems II.1a, b, c, d, e. Ignoring edge effects, the expected number of penetrators killed is given by

$$E(\Delta \tilde{B}) = B \left[1 - \prod_{j=1}^n (1-q_j)^{R_j} \right],$$

where the q_j are defined separately for each attack protocol (a) through (e), as follows.

- Attack protocol (a)

$$q_j = k_j w_j d_j / L$$

- Attack protocol (b)

If $H_j = 0$, $q_j = 0$.

If $H_j \geq 1$ and $I_j = 0$,

$$q_j = [1 - (1-k_j)^{H_j}] \frac{w_j d_j}{L}.$$

If $H_j \geq 1$ and $I_j \geq 1$

$$q_j = \frac{w_j d_j}{L} - \sum_{i=0}^{I_j} \theta_{ji} \left(1 - \frac{k_j}{i+1}\right)^{H_j}.$$

- Attack protocol (c)

If $H_j = 0$, $q_j = 0$.

If $H_j \geq 1$ and $I_j = 0$,

$$q_j = [1 - (1-k_j)^{H_j}] \frac{w_j d_j}{L},$$

as in protocol (b).

If $H_j \geq 1$ and $I_j \geq 1$

$$q_j = \sum_{i=0}^{I_j} [1 - (1-k_j) \left\lfloor \frac{H_j}{i+1} \right\rfloor (1 - \langle \frac{H_j}{i+1} \rangle_{k_j})] \theta_{ji} .$$

- Attack protocol (d)

If $H_j = 0$, $q_j = 0$.

If $1 \leq H_j \leq I_j$

$$\begin{aligned} q_j = & \frac{k_j}{B} I_j (1-f_j) d_j \mathbb{B}(H_j-1; I_j-1, d_j) \\ & + \frac{k_j}{B} (I_j+1) f_j d_j \mathbb{B}(H_j-1; I_j, d_j) \\ & + \frac{k_j}{B} H_j (1-f_j) [1 - \mathbb{B}(H_j; I_j, d_j)] \\ & + \frac{k_j}{B} H_j f_j [1 - \mathbb{B}(H_j; I_j+1, d_j)] . \end{aligned}$$

If $I_j = 0$ or $H_j \geq I_j + 1$, $q_j = k_j w_j d_j / L$.

- Attack protocol (e)

$$q_j = \frac{k_j}{B} [1 - (1-d_j)^{I_j} (1-f_j d_j)] .$$

If $I_j = 0$, this reduces to

$$q_j = k_j w_j d_j / L .$$

The above results for attack protocols (b) and (c) do not, in general, have closed forms. If the detection probability d_j equals 1, however, most of the θ_{ji} are zero and the formulas for q_j are easily evaluated. In the implementation of these protocols in the NAVMOD model [3], parameters w_j and d_j are input but the code evaluates the expression for q_j using $w_j d_j$ and 1 instead of w_j and d_j .

It can be verified that if H_j is equal to 1, the formulas for q_j in attack protocols (b), (c), and (d) all reduce to the corresponding formulas in protocol (e).

The expected number of shots fired at a specific barrier element is given by

Theorems II.2a, b, c, d, e: Ignoring edge effects, the expectation $E(\tilde{y})$ of the number of shots \tilde{y} fired at a particular barrier element β by the penetrators is

$$E(\tilde{y}) = \sum_{j=1}^n R_j e_j ,$$

where the e_j are defined separately for each attack protocol, as follows.

- Attack protocol (a)

$$e_j = w_j d_j / L .$$

- Attack protocol (b)

$$e_j = \frac{H_j}{B} [1 - (1-d_j)^{I_j} (1-f_j d_j)] .$$

- Attack protocol (c)

$$e_j = \frac{H_j}{B} [1 - (1-d_j)^{I_j} (1-f_j d_j)] .$$

- Attack protocol (d)

$$e_j = q_j / k_j ,$$

where q_j is as defined in Theorem II.1d, for the appropriate combination of H_j and I_j values.

- Attack protocol (e)

$$e_j = \frac{1}{B} [1 - (1-d_j)^{I_j} (1-f_j d_j)] .$$

In accordance with the arguments of Section C.2, the probability distribution of \tilde{y} , the total number of shots fired at a given barrier element, is given for the following cases.

- There is only one penetrator type ($n=1$) and attack protocol (a), (d), or (e) is used.
- There is only one penetrator type, attack protocol (b) or (c) is used; and $w_1 \leq L/B$.
- There are multiple penetrator types and attack protocol (a), (d), or (e) is used.

Theorems II.3a, d, e: Let $n=1$. Then, ignoring edge effects, \tilde{y} , the number of shots fired at a particular barrier element, is binomially distributed with parameters R and ϕ_{11} , where ϕ_{11} is defined separately for attack protocols (a), (d), and (e), as follows.

- Attack protocol (a)

$$\phi_{11} = w_1 d_1 / L$$

- Attack protocol (d)

If $H_1 = 0$, $\phi_{11} = 0$.

If $H_1 \geq 1$ and $I_1 = 0$

$$\phi_{11} = w_1 d_1 / L$$

If $H_1 \geq 1$ and $I_1 \geq 1$

$$\begin{aligned} \phi_{11} = & \frac{1}{B} I_1 (1-f_1) d_1 \mathbb{B}(H_1-1; I_1-1, d_1) \\ & + \frac{1}{B} (I_1+1) f_1 d_1 \mathbb{B}(H_1-1; I_1, d_1) \\ & + \frac{1}{B} H_1 (1-f_1) [1 - \mathbb{B}(H_1; I_1, d_1)] \\ & + \frac{1}{B} H_1 f_1 [1 - \mathbb{B}(H_1; I_1+1, d_1)] . \end{aligned}$$

(This is identical to e_1 and q_1/k_1 .) It can be verified that if $H_1 \geq I_1 + 1$, the above formula reduces to

$$\phi_{11} = w_1 d_1 / L .$$

- Attack protocol (e)

$$\phi_{11} = \frac{1}{B} [1 - (1-d_1)^{I_1} (1-f_1 d_1)] .$$

The subscripts on ϕ_{11} are not really necessary; we have kept them because ϕ_{11} is the same quantity indicated by that symbol in Section II.C, above: the probability a given penetrator fires exactly one shot at the particular barrier element considered.

Theorems II.3b and c: Suppose that $n=1$ and that attack protocol (b) or (c) is used. If $w_1 \leq L/B$, then

$$P(\tilde{y} = mH_1) = \binom{R}{m} \left(\frac{w_1 d_1}{L} \right)^m \left(1 - \frac{w_1 d_1}{L} \right)^{R-m}, \quad m=0, \dots, R,$$

and $P(\tilde{y}=y) = 0$ for all other values of y . (I.e., \tilde{y}/H_1 is binomially distributed with parameters R and $w_1 d_1/L$.)

Theorems II.4a, d, e: Suppose that attack protocol (a), (d), or (e) is used. Then, possibly subject to edge effects, \tilde{y} is approximately Poisson distributed with mean

$$\mu = \sum_{j=1}^n R_j e_j, \quad ,$$

where the e_j are as given in Theorems II.2a, d, and e above.

(Since in attack protocols (a), (d), and (e), a shooter fires no more than one shot at any particular target, the quantity e_j , which is the expected number of shots fired at a given target by a given shooter, is the same as the probability the shooter fires one shot at that target.)

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APPENDIX A
PROOFS OF RESULTS IN CHAPTER I

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PROOFS OF RESULTS IN CHAPTER I

This appendix is in two parts. The first part proves seven lemmas that are used in deriving the overall formulas. The first four were stated in Chapter I, the others provide useful algebraic results. The lemmas involve the following issues:

Lemma I.1--Independence condition for attack protocol (a).

Lemma I.2--Independence condition if $w_j \leq L/B$.

Lemma I.3--Formula for $N_j(x)$.

Lemma I.4--Formula for $P(\tilde{y}=y)$, in terms of $P(F_y)$ when $w_j \leq L/B$.

Lemma I.5--Derivation of probabilities $P(F_h|DA_1)$.

Lemma I.6--Formulas for $P(K|DA_1)$ and $E(\tilde{h}|DA_1)$.

Lemma I.7--Formulas for sums and weighted sums of ω_l and ψ_{jl} .

The second part of the appendix presents the steps in proving the theorems of Chapter I, Section G. The main conditioning arguments have been presented in Chapter I, Section E. Here, these will be briefly restated, with additional arguments as necessary to complete the proofs.

Unless specifically redefined, the notation used in this appendix matches that of Chapter I. The parameters R_j , R , B , n , L , w_j , d_j , k_j , and H , and the attack protocols (a) through (e) are as defined in Chapter I, Section A. We let I_j and f_j denote the integer and fractional parts, respectively, of $w_j B/L$. The events and random variables F_h , D , DA_1 , K , and \tilde{h} concerning one penetrator and one barrier element are as defined in Chapter I, Section E.1. The events and random variables S , S_b , C_x , \tilde{x} , $N_j(\tilde{x})$, \tilde{y}_b , and \tilde{y} are as defined in Chapter I, Section E.2.

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Lemma I.5--Derivation of probabilities $P(F_h|DA_t)$.

Lemma I.6--Formulas for $P(K|DA_t)$ and $E(\tilde{h}|DA_t)$.

Lemma I.7--Formulas for sums and weighted sums of ω_t and ψ_{jt} .

The second part of the appendix presents the steps in proving the theorems of Chapter I, Section G. The main conditioning arguments have been presented in Chapter I, Section E. Here, these will be briefly restated, with additional arguments as necessary to complete the proofs.

Unless specifically redefined, the notation used in this appendix matches that of Chapter I. The parameters R_j , R , B , n , L , w_j , d_j , k_j , and H , and the attack protocols (a) through (e) are as defined in Chapter I, Section A. We let I_j and f_j denote the integer and fractional parts, respectively, of $w_j B/L$. The events and random variables F_h , D , DA_t , K , and \tilde{h} concerning one penetrator and one barrier element are as defined in Chapter I, Section E.1. The events and random variables S , S_b , C_x , \tilde{x} , $N_j(\tilde{x})$, \tilde{y}_b , and \tilde{y} are as defined in Chapter I, Section E.2.

A. THE LEMMAS

Lemma I.1 Consider one penetrator ρ , of type j . Let the events S_b , S , and C_x be as defined in Chapter I, Section E.2. Suppose that attack protocol (a) is used. Then, for every $x \in [0, L]$

$$P(S|C_x) = \prod_{b=1}^B P(S_b|C_x) .$$

Proof: Suppose that $N_j(x)$ barrier elements can potentially detect the penetrator if the penetrator crosses at point x . For now, let N denote $N_j(x)$. By Assumption 4), the number \tilde{m} of barrier elements that actually detect the penetrator is binomially distributed with parameters N and d_j . Since attack protocol (a) is used, \tilde{m} is also the number of shots fired at penetrator ρ . By Assumption 7), the probability ρ survives, given that m shots are fired at it, is $(1-k_j)^m$, regardless of ρ 's crossing point. Then

$$P(S|C_x) = \sum_{m=0}^N (1-k_j)^m \binom{N}{m} d_j^m (1-d_j)^{N-m} .$$

Using the formula for the z-transform of the binomial¹ this becomes

$$P(S|C_x) = [1-d_j + d_j(1-k_j)]^N = (1-d_j k_j)^{N_j(x)} .$$

Now consider barrier element b . Whether b can detect ρ depends on ρ 's crossing point x . Given x , if b cannot potentially detect ρ , ρ will certainly survive b and

$$P(S_b|C_x) = 1 .$$

If b can potentially detect ρ , then b will actually detect ρ with probability d_j . Since protocol (a) is used, b will then fire one shot at ρ , which will be lethal with probability k_j . Thus if b can potentially detect a penetrator crossing at x

$$P(S_b|C_x) = 1 - d_j k_j .$$

Since given C_x , $N_j(x)$ penetrators can potentially detect ρ ,

¹ $\sum_{m=0}^n z^m b(m;n,p) = (1-p+pz)^n$

where the notation of Chapter I, Section G for binomial probabilities has been used.

$$\prod_{b=1}^B P(S_b|C_x) = (1 - d_j k_j)^{N_j(x)},$$

which equals the previously derived formula for $P(S|C_x)$.

Lemma I.2 If penetrator ρ is of type j and $w_j \leq L/B$, then the events S_b that ρ survives barrier elements b , for $b=1, \dots, B$, are mutually independent given the crossing point x of ρ .

Proof: Let the events S and C_x be as defined in Chapter I, Section E.2. By Lemma I.3, below, the number of barrier elements that can potentially detect ρ is I_j with probability $1 - f_j$ and $I_j + 1$ with probability f_j . If $w_j \leq L/B$, either $I_j = 0$ or $I_j = 1$ and $f_j = 0$, so that with probability one ρ is vulnerable to at most one barrier element. If x is such that ρ is vulnerable to no barrier element, then ρ survives each barrier element, so

$$P(S|C_x) = 1 = \prod_{b=1}^B P(S_b|C_x).$$

If x is such that ρ is vulnerable to one barrier element β , then ρ survives overall if and only if it survives β , so $P(S_\beta|C_x) = P(S|C_x)$. For $b \neq \beta$, ρ is not vulnerable to barrier element b , thus $P(S_b|C_x) = 1$. Thus

$$P(S|C_x) = \prod_{b=1}^B P(S_b|C_x)$$

as stated.

Lemma I.3 Let the random variable \tilde{x} be the crossing point of some specific penetrator ρ of type j . Let $N_j(\tilde{x})$ be the number of barrier elements that can potentially detect that penetrator. Then, ignoring edge effects,

$$N_j(\tilde{x}) = \begin{cases} I_j & \text{w.p. } 1-f_j \\ I_j + 1 & \text{w.p. } f_j \end{cases}$$

where $I_j = \left\lfloor \frac{w_j B}{L} \right\rfloor$ and $f_j = \frac{w_j B}{L} - I_j$. Alternatively stated, the set of $x \in [0, L]$ where $N_j(x) = I_j$ has length $(1-f_j)L$, and the set of $x \in [0, L]$ where $N_j(x) = I_j + 1$ has length $f_j L$.

Proof: Let the barrier line (of length L , by Assumption 1)) be labeled with coordinates 0 through L . The number of barrier elements, B , is assumed to be an integer ≥ 1 . By the positioning described in Assumption 1), the coordinate of barrier element b is

$$x_b = \frac{L}{B} (b - \frac{1}{2}) , \quad b=1, \dots, B.$$

By Assumption 2), \tilde{x} is distributed uniformly on $[0, L]$. For any specific crossing point $x \in [0, L]$ of penetrator ρ , by Assumption 4), barrier element b will detect penetrator ρ if and only if

$$|x_b - x| \leq w_j/2 .$$

Substituting $\frac{L}{B} (b - \frac{1}{2})$ for x_b , denoting w_j/L by θ , and performing the algebra yields the equivalent condition

$$\frac{1}{2} + B (\frac{x}{L} - \frac{\theta}{2}) \leq b \leq \frac{1}{2} + B (\frac{x}{L} - \frac{\theta}{2}) + B\theta .$$

It is clear that $N_j(x)$ is the cardinality of the set of b that satisfy the above condition. We ignore edge effects by letting b assume any integer values, possibly nonpositive or exceeding B . (That is, ρ crosses in the middle of an infinitely long barrier.) Denote the left-hand side of the above expression by $b_0(x)$, i.e., for each $x \in [0, L]$ let

$$b_0(x) = \frac{1}{2} + B (\frac{x}{L} - \frac{\theta}{2}) .$$

Denote $(1-B\theta)/2$ by a_0 . Then, since $\tilde{x} \sim U[0, L]$, $b_0(\tilde{x})$ is a random variable uniformly distributed on $[a_0, a_0+B]$, and is integer with probability zero. The desired quantity $N_j(\tilde{x})$ is the number of integer points in the interval

$$[b_0(\tilde{x}) , b_0(\tilde{x}) + B\theta] .$$

To determine this number, let us simplify the notation. Let $b_0(\tilde{x})$ be denoted by \tilde{a} . For any noninteger a and any nonnegative y , the integer points in the interval $[a, a+y]$ are $\lfloor a \rfloor + 1, \lfloor a \rfloor + 2, \dots$, up to $\lfloor a \rfloor + k$, where k is the greatest integer such that

$$\lfloor a \rfloor + k \leq a + y .$$

Rearranging, we see that k , which is the number of integer points in the interval, equals

$$\lfloor (a+y - \lfloor a \rfloor) \rfloor ,$$

which equals

$$\lfloor y \rfloor + \lfloor \langle a \rangle + \langle y \rangle \rfloor .$$

Since fractional parts $\langle a \rangle$ and $\langle y \rangle$ are both in the interval $[0,1)$,

$$\lfloor \langle a \rangle + \langle y \rangle \rfloor = \begin{cases} 0 & \text{if } \langle a \rangle + \langle y \rangle < 1 \\ 1 & \text{if } \langle a \rangle + \langle y \rangle \geq 1 \end{cases} .$$

Now let \tilde{a} be a random variable distributed uniformly on $[a_0, a_0 + B]$. Remember that B is the number of barrier elements, a positive integer; y is a nonnegative constant. Define the random variable

$$\tilde{k} = \lfloor y \rfloor + \lfloor \langle \tilde{a} \rangle + \langle y \rangle \rfloor .$$

The interval $[a_0, a_0 + B]$ is composed of B unit intervals. As \tilde{a} ranges over the interval $[a_0, a_0 + 1)$, $\langle \tilde{a} \rangle$ assumes every value between 0 and 1 (except for 1 itself). The same is true as \tilde{a} ranges over

$$[a_0 + 1, a_0 + 2), [a_0 + 2, a_0 + 3), \dots, [a_0 + (B-1), a_0 + B) .$$

Therefore $\langle \tilde{a} \rangle$ is uniformly distributed on $[0,1]$ and is thus less than $1 - \langle y \rangle$ with probability $1 - \langle y \rangle$. Then the distribution of \tilde{k} is

$$\tilde{k} = \begin{cases} \lfloor y \rfloor & \text{w.p. } 1 - \langle y \rangle \\ \lfloor y \rfloor + 1 & \text{w.p. } \langle y \rangle . \end{cases}$$

But as it has been defined, \tilde{k} is in fact $N_j(\tilde{x})$. Let $y = B\theta = w_j B/L$. Substituting these in the above expression yields the statement of the lemma.

We note that if $w_j \leq L/B$, then for every $x \in [0, L]$ the condition

$$\frac{1}{2} + B\left(\frac{x}{L} - \frac{\theta}{2}\right) \leq b \leq \frac{1}{2} + B\left(\frac{x}{L} - \frac{\theta}{2}\right) + B\theta$$

for barrier elements b that can potentially detect a penetrator crossing at x strictly includes at most one integer value of b , and this value is not negative or strictly greater than B . This is true because $w_j \leq L/B$ if and only if $B\theta \leq 1$. The interval in which b can lie, therefore, has width ≤ 1 . Its left endpoint is greater than or equal to Bx/L , which is never negative

for $x \in [0, L]$. Similarly, the right endpoint is $\leq Bx/L + 1$, so the interval includes the integer point $B + 1$ only if $x = L$, which happens with probability zero. Thus, if $w_j \leq L/B$, the edge effect of "detection" of a penetrator by mythical barrier elements will not occur.

Lemma I.4 Suppose $w_i \leq L/B$ for all $i = 1, \dots, n$. Then for all attack protocols

- If $w_j < L/B$

$$P(\tilde{y}=0) = 1 - f_j + f_j P(F_0)$$

$$P(\tilde{y}=y) = f_j P(F_y), \quad y=1, \dots, H,$$

- If $w_j = L/B$

$$P(\tilde{y}=y) = P(F_y), \quad y=0, \dots, H,$$

where in attack protocols (a) and (e) H is taken to be 1.

Proof: Let \tilde{y}_b be the number of shots fired at the given penetrator ρ (of type j) by barrier element b , so \tilde{y} , the total number of shots fired at ρ , is

$$\tilde{y} = \sum_{b=1}^B \tilde{y}_b.$$

F_y is the event that a given barrier element β fires exactly y shots at a given penetrator ρ (of type j). The probabilities $P(F_y)$ are calculated assuming that β can potentially detect ρ . Let \tilde{N} denote the number of barrier elements that can potentially detect ρ . By Lemma I.3,

$$\tilde{N} = \begin{cases} I_j & \text{w.p. } 1-f_j \\ I_j+1 & \text{w.p. } f_j \end{cases},$$

and if $w_j \leq L/B$, edge effects are not a problem. A barrier element that cannot potentially detect ρ fires zero shots at ρ . Thus $P(\tilde{y}=0 \mid \tilde{N}=0) = 1$ and for $y=1, 2, \dots$, $P(\tilde{y}=y \mid \tilde{N}=0) = 0$. Also $P(\tilde{y}=y \mid \tilde{N}=1)$ is simply $P(F_y)$, for $y=0, 1, 2, \dots$, because if $\tilde{N} = 1$, the number of shots ρ receives overall is simply the number of shots ρ receives from the one barrier element that can potentially detect ρ .

Always,

$$P(\tilde{y}=y) = P(\tilde{y}=y | \tilde{N}=I_j)(1-f_j) + P(\tilde{y}=y | \tilde{N}=I_j+1) f_j.$$

If $w_j < L/B$, $I_j = 0$, and if $w_j = L/B$, $I_j = 1$ and $f_j = 0$. Making the appropriate substitutions for the indicated conditional probabilities in the above equation yields the statements in the lemma. (Recall from Chapter I, Section E that $P(F_y) = 0$ for $y > H$.)

Lemma I.5 Consider one attacker, barrier element β , and one target, penetrator ρ . Assume that β has detected ρ and exactly ι other targets. Then the probability $P(F_h | DA_\iota)$ that β fires exactly h shots at ρ is given by the following formulas, for each attack protocol in turn (ι can be any nonnegative integer).

- Attack protocol (a)

$$P(F_1 | DA_\iota) = 1$$

$$P(F_h | DA_\iota) = 0 \text{ for all other } h.$$

- Attack protocol (b)

$$P(F_h | DA_\iota) = \binom{H}{h} \left(\frac{1}{\iota+1}\right)^h \left(1 - \frac{1}{\iota+1}\right)^{H-h}, \quad h=0, \dots, H.$$

- Attack protocol (c) Let J denote the integer part and η the fractional part of $\frac{H}{\iota+1}$. Then

$$P(F_J | DA_\iota) = 1 - \eta$$

$$P(F_{J+1} | DA_\iota) = \eta$$

$$P(F_h | DA_\iota) = 0 \text{ for all other } h.$$

- Attack protocol (d).

$$P(F_0 | DA_\iota) = 1 - \min\left(1, \frac{H}{\iota+1}\right)$$

$$P(F_1 | DA_\iota) = \min\left(1, \frac{H}{\iota+1}\right)$$

$$P(F_h | DA_\iota) = 0 \quad h \geq 2.$$

- Attack protocol (e)

$$P(F_0 | DA_1) = 1 - \frac{1}{t+1}$$

$$P(F_1 | DA_1) = \frac{1}{t+1}$$

$$P(F_h | DA_1) = 0, \quad h \geq 2.$$

Proof: In all cases, the results follow from the statement of the attack protocols in Assumption 5) of the problem statement in Chapter I, Section A, where the total number of targets β detects is $t + 1$. The formulas for attack protocols (a) and (e) are self-explanatory. The formula for (b) is a precise statement of the attack protocol. In protocol (c), each target will receive J or $J + 1$ shots, and J shots at each target uses up $J(t+1)$ shots, so there are $H - J(t+1)$ shots left over, to be distributed among the $t + 1$ targets such that no target gets more than one of the additional shots. The probability that a target does get one of the additional shots can be seen to be

$$\frac{H - J(t+1)}{t + 1} = \frac{H}{t+1} - J = \eta.$$

For attack protocol (d), if $H \geq t + 1$, then each target will receive one shot, i.e., $P(F_1 | DA_1) = 1$. If $H < t + 1$, there are H shots to be apportioned among $t + 1$ targets such that no target gets more than one shot. The probability that β gets one of the shots is simply the number of subsets of the $t + 1$ targets of size H that contain β , divided by the total number of subsets of size H of the $t + 1$ targets, i.e.,

$$P(F_1 | DA_1) = \frac{\binom{t}{H-1}}{\binom{t+1}{H}}$$

which simplifies to $\frac{H}{t+1}$. Combining the two results, we obtain, as stated

$$P(F_1 | DA_1) = \min(1, \frac{H}{t+1}).$$

Lemma I.6 Let the probabilities $P(F_h | DA_1)$ be as given in Lemma I.5. Then the quantities

$$P(K|DA_{\iota}) = \sum_{h=0}^H (1-[1-k_j]^h) P(F_h|DA_{\iota})$$

and

$$E(\tilde{h}|DA_{\iota}) = \sum_{h=0}^H h P(F_h|DA_{\iota})$$

are as given in Table A-1. (Note that ι can be any non-negative integer. In the homogeneous case, substitute k for k_j .)

Proof: The only issue is the algebra, which is straightforward except for the evaluation of $P(K|DA_{\iota})$ in attack protocol (b). There we use the formula for the z-transform of the binomial

$$\sum_{m=0}^n z^m b(m; n, p) = (1-p+pz)^n.$$

In protocol (c), note that $\left\lfloor \frac{H}{\iota+1} \right\rfloor + \left\langle \frac{H}{\iota+1} \right\rangle = \frac{H}{\iota+1}$; this simplifies the formula for

$E(\tilde{h}|DA_{\iota})$. The formulas for $E(\tilde{h}|DA_{\iota})$ are identical for attack protocols (b) and (c). This is reasonable given the nature of the protocols. In (b), an attacker scatters its H shots randomly over the $\iota + 1$ targets it has detected; in (c) it distributes the shots as evenly as

possible, but in either case there is an average of $\frac{H}{\iota+1}$ shots per detected target. Since the

$E(\tilde{h}|DA_{\iota})$ are the same for all ι , the overall $E(\tilde{h})$ and $E(\tilde{y})$ are the same for protocols (b) and (c).

Table A-1. FORMULAS FOR $P(K|DA_i)$ AND $E(\tilde{h}|DA_i)$

Attack Protocol	$P(K DA_i)$	$E(\tilde{h} DA_i)$
(a)	k_j	1
(b)	$1 - \left(1 - \frac{k_j}{i+1}\right)^H$	$\frac{H}{i+1}$
(c)	$1 - (1-k_j)^{\left\lfloor \frac{H}{i+1} \right\rfloor} (1 - \langle \frac{H}{i+1} \rangle k_j)$	$\frac{H}{i+1}$
(d)	$k_j \min\left(\frac{H}{i+1}, 1\right)$	$\min\left(\frac{H}{i+1}, 1\right)$
(e)	$\frac{k_j}{i+1}$	$\frac{1}{i+1}$

Lemma I.7 Let the notation b , B , p , and P for binomial and Poisson probabilities be as in Chapter I, Section G. Let R_j , R , d , d_j , w , w_j , and L be as in Chapter I, Section A; assume that they obey the restrictions in Chapter I, Section B. As in Chapter I, Section G, let $p = dw/L$, and define

$$\omega_i = d \binom{R-1}{i} p^i (1-p)^{R-1-i}, \quad i=0, \dots, R-1$$

and

$$\psi_{ji} = d_j e^{-\alpha_j} \alpha_j^i / i!, \quad i=0, 1, 2, \dots,$$

where, for the purposes of this lemma, the α_j are some positive constants. (The index j is redundant here.) Then for integers i_1 and i_2 such that $0 \leq i_1 < i_2$:

$$a) \quad \sum_{i=i_1}^{i_2} \omega_i = d[B(i_2; R-1, p) - B(i_1-1; R-1, p)],$$

$$\begin{aligned}
\text{b)} \quad & \sum_{i=i_1}^{i_2} \frac{1}{i+1} \omega_i = \frac{L}{wR} [\mathbb{E}(i_2+1; R, p) - \mathbb{E}(i_1; R, p)], \\
\text{c)} \quad & \sum_{i=i_1}^{i_2} \psi_{ji} = d_j [\mathbb{P}(i_2; \alpha_j) - \mathbb{P}(i_1-1; \alpha_j)], \\
\text{d)} \quad & \sum_{i=i_1}^{i_2} \frac{1}{i+1} \psi_{ji} = \frac{d_j}{\alpha_j} [\mathbb{P}(i_2+1; \alpha_j) - \mathbb{P}(i_1; \alpha_j)].
\end{aligned}$$

Proof: It is evident that $\omega_i = d \mathbb{b}(i; R-1, p)$ and $\psi_{ji} = d_j \mathbb{p}(i; \alpha_j)$ so formulas a) and c) follow immediately from the definitions of $\mathbb{B}(m; n, p)$ and $\mathbb{P}(m; \mu)$ in Chapter I, Section G. Note also that for $0 \leq i \leq R-1$

$$\frac{1}{i+1} \omega_i = d \frac{(R-1)!}{(R-1-i)! i! (i+1)} p^i (1-p)^{R-1-i}.$$

Multiplying and dividing the right-hand side by Rp and simplifying yields

$$\frac{1}{i+1} \omega_i = \frac{d}{Rp} \mathbb{b}(i+1; R, p)$$

and thus

$$\begin{aligned}
\sum_{i=R_1}^{i_2} \frac{1}{i+1} \omega_i &= \sum_{m=i_1+1}^{i_2+1} \frac{d}{Rp} \mathbb{b}(m; R, p) \\
&= \frac{d}{Rp} [\mathbb{B}(i_2+1; R, p) - \mathbb{B}(i_1; R, p)].
\end{aligned}$$

Substituting wd/L for p yields expression b). If $i_2 \geq R$, $\mathbb{B}(i_2+1; R, p)$ remains 1, and thus expression b) holds for all values of i_1 and i_2 .

Similarly

$$\frac{1}{i+1} \psi_{ji} = d_j e^{-\alpha_j} \alpha_j^i / (i+1)! = \frac{d_j}{\alpha_j} \mathbb{p}(i+1; \alpha_j).$$

Performing the indicated sum clearly yields expression d).

B. PROOFS OF THEOREMS IN CHAPTER I, SECTION G

This section presents outlines for proofs of the theorems stated in Chapter I, Section G. The main conditioning arguments have been derived in Chapter I, Section E. Here, we briefly restate them as appropriate; see Section E for justifications. Actual algebraic simplifications are not performed, but references to lemmas are made where appropriate. Unless specifically redefined, all symbols for parameters, events, random variables, and so forth, are as in Chapter I.

Homogeneous ($n=1$) and heterogeneous ($n>1$) cases are treated together: the derivations are very similar. In the homogeneous case, suppress the subscripts j on the parameters, and consider ω_1 as being zero for $i \geq R$.

In the heterogeneous case where some α_j are zero, some results have a form different from the general equations. In the proofs of the theorems we assume $\alpha_j > 0$; in Section 4, below, we discuss the case $\alpha_j = 0$.

Independence conditions, Poisson approximations, and edge effects are not mentioned in this appendix; they affect the final results in the ways discussed in Chapter I, Sections D, E, and F.

For the rest of this appendix, "Section" refers to the indicated section in Chapter I.

1. Theorems I.1 and I.2: Expected Numbers of Penetrators of Type j Killed

The derivation starts by evaluating the probability $P(K)$ (that a specific barrier element kills a specific penetrator which it can potentially detect), using the formula from Section E.1 (page 19)

$$P(K) = \sum_{h=1}^H \sum_{i=0}^{\infty} P(K|F_h) P(F_h|DA_i) P(DA_i).$$

The $P(F_h|DA_i)$ are given by Lemma I.5, $P(K|F_h)$ is simply $1-(1-k_j)^h$, and the $P(DA_i)$ are the ω_i or (approximately) ψ_{ji} defined in Section G.1. To evaluate $P(K)$, first interchange the order of summation. The sum on h

$$\sum_{h=1}^H P(K | F_h) P(F_h | DA_1)$$

results in the $P(K | DA_1)$ given in Lemma I.6. Then

$$P(K) = \sum_{i=0}^{\infty} P(K | DA_i) P(DA_i).$$

The results of Lemma I.7 can be used to evaluate this sum for attack protocols (a), (d), and

(e). In (d), the sum must be broken into two parts: where $P(K | DA_1)$ is k_j or $\frac{k_j H}{i+1}$, respectively. The sum for protocol (b) does not appear to have a closed form. In

evaluating the sum for (c), we note that for $i \geq H$, the integer and fractional parts of $\frac{H}{i+1}$

become zero and $\frac{H}{i+1}$, and thus the $P(K | DA_i)$ of Table A-1 becomes

$$P(K | DA_i) = \frac{kH}{i+1}.$$

The tail sum

$$\sum_{i=H}^{\infty} \frac{kH}{i+1} P(DA_i)$$

can be evaluated using Lemma I.7 (noting that in the homogeneous case $P(DA_1) = 0$ for $i \geq R$, and in the heterogeneous case, letting $P(\infty; \mu)$ equal 1 for any μ). The rest of the sum

$$\sum_{i=0}^{H-1} P(K | DA_i) P(DA_i)$$

for protocol (c) does not seem to have a closed form.

The resultant $P(K)$ for protocols (b), (c), and (e) are stated in Theorems I.1 and I.2; for protocol (a), $P(K) = d_j k_j$. To derive the overall $E(\bar{\Delta R}_j)$, we first note that

$$E(\Delta R_j) = R_j P(S)$$

(Reference [12]), and also

$$\begin{aligned}
 P(\bar{S}) &= 1 - P(S) \\
 P(S) &= \int_0^L P(S | C_x) \frac{1}{L} dx \\
 P(S | C_x) &= \prod_{b=1}^B P(S_b | C_x) \text{ if the independence condition holds} \\
 &= [1 - P(K)]^{N_j(x)}.
 \end{aligned}$$

Using Lemma I.3, the integration on x yields

$$\begin{aligned}
 P(S) &= [1 - P(K)]^{I_j} (1 - f_j) + [1 - P(K)]^{I_j + 1} f_j \\
 &= [1 - P(K)]^{I_j} [1 - f_j P(K)]
 \end{aligned}$$

and the resultant formula for $E(\Delta R_j)$ is as displayed in Theorems I.1 and I.2.

2. Theorems I.3 and I.4: The Expected Number of Shots Fired at a Given Penetrator

First, consider one specific penetrator (of type j) and find the expected value of the number of shots \tilde{h} fired at the penetrator by a given barrier element that can potentially detect it. By arguments in Section E.1,

$$E(\tilde{h}) = \sum_{h=1}^H \sum_{t=0}^{\infty} h P(F_h | DA_t) P(DA_t).$$

The $P(F | DA_t)$ are given by Lemma I.5; the $P(DA_t)$ are ω_t or ψ_{jt} for the homogeneous and heterogeneous cases, respectively. Interchange the order of summation; the intermediate sums

$$E(\tilde{h} | DA_t) = \sum_{h=1}^H h P(F_h | DA_t)$$

are derived in Lemma I.6. Examining these results, it is evident that for all attack protocols the sums

$$E(\tilde{h}) = \sum_{t=0}^{\infty} E(\tilde{h} | DA_t) P(DA_t)$$

can be evaluated using Lemma I.7. The results are given in Table A-2.

Table A-2. EXPRESSIONS FOR $E(\tilde{h})$

Attack Protocol	Homogeneous Case	Heterogeneous Case
(a)	d	d_j
(b)	$\frac{LH}{wR} [1 - (1 - \frac{wd}{L})^R]$	$\frac{Hd_j}{\alpha_j} (1 - e^{-\alpha_j})$
(c)	$\frac{LH}{wR} [1 - (1 - \frac{wd}{L})^R]$	$\frac{Hd_j}{\alpha_j} (1 - e^{-\alpha_j})$
(d)	<p>If $H \geq R$, $E(\tilde{h}) = d$</p> <p>If $H \leq R-1$</p> $E(\tilde{h}) = d \mathbb{B}(H-1; R-1, p) + \frac{HL}{wR} [1 - \mathbb{B}(H; R, p)]$ <p>where $p = wd/L$.</p>	$d_j \mathbb{P}(H-1; \alpha_j) + \frac{Hd_j}{\alpha_j} [1 - \mathbb{P}(H; \alpha_j)]$
(e)	$\frac{L}{wR} [1 - (1 - \frac{wd}{L})^R]$	$\frac{d_j}{\alpha_j} (1 - e^{-\alpha_j})$

To compute $E(\tilde{y})$, the expected number of shots fired at a given penetrator, ρ , we note that

$$E(\tilde{y}) = \sum_{b=1}^B E(\tilde{y}_b) .$$

where \tilde{y}_b is the expected number of shots fired at the penetrator by barrier element b . If b can potentially detect ρ , \tilde{y}_b has the same distribution as \tilde{h} ; if b cannot potentially detect ρ , \tilde{y}_b is zero with certainty. Given that ρ crosses at point x , $N_j(x)$ barrier elements can potentially detect ρ . Thus

$$\begin{aligned} E(\tilde{y}|C_x) &= \sum_{b=1}^B E(\tilde{y}_b|C_x) \\ &= N_j(x) E(\tilde{h}) + (B - N_j(x)) \cdot 0 . \end{aligned}$$

Integrating on x and using Lemma I.3,

$$\begin{aligned} E(\tilde{y}) &= \int_0^L E(\tilde{y}|C_x) \frac{1}{L} dx \\ &= (1-f_j) I_j E(\tilde{h}) + f_j (I_j + 1) E(\tilde{h}) . \end{aligned}$$

Simplifying and recalling that (by definition) $I_j + f_j = w_j B/L$,

$$E(\tilde{y}) = \frac{w_j B}{L} E(\tilde{h}) .$$

In the homogeneous case, the w_j and L will cancel terms from $E(\tilde{h})$. The final results for $E(\tilde{y})$ are as stated in Theorems I.3 and I.4.

3. Theorems I.5 and I.6: The Probability Distribution of the Number of Shots Fired at a Given Penetrator

The proof of Theorems I.5a and I.6a (when attack protocol (a) is used) has essentially been given in Section E.2. We note here that Lemma I.3 is used in integrating over x . The algebraic result for $P(\tilde{y}=y)$ in that section is the same as the result stated in Theorem I.6a. Note that if $I_j=0$, $P(\tilde{y}=0) = 1-f_j d_j$ and $P(\tilde{y}) = f_j d_j$. This makes sense: a penetrator receives one shot if and only if it is detectable by some attacker (which occurs with probability f_j) and that attacker then actually detects the penetrator.

For the other attack protocols, we assume that $w_j \leq L/B$ for all $j=1, \dots, n$. Lemma I.4 reduces the problem to finding the probability $P(F_y)$ that a specific barrier element β fires exactly y shots at a specific penetrator ρ that the barrier element can potentially detect. The quantity y can range from zero through H , the input number of shots per shooter. In attack protocol (e), set $H=1$. The formula in Section E.1

$$P(F_h) = P(F_h | \bar{D}) P(\bar{D}) + \sum_{i=0}^{\infty} P(F_h | DA_i) P(DA_i)$$

yields

$$P(F_0) = 1 - d_j + \sum_{i=0}^{\infty} P(F_0 | DA_i) P(DA_i)$$

and

$$P(F_y) = \sum_{i=0}^{\infty} P(F_y | DA_i) P(DA_i), \quad y=1, \dots, H,$$

using y rather than h as an index. The $P(F_y | DA_i)$ are given in Lemma I.5; the $P(DA_i)$ are the ω_i or ψ_{ji} defined in Section G.1. Finding the $P(F_y)$ then becomes a matter of evaluating the indicated sums above. Looking at the Lemma I.5 results, we see that the results of Lemma I.7 can be applied in a straightforward manner to attack protocols (d) and (e) for the case $y=1$. Since in these protocols, a given barrier element will never fire more than one shot at a given penetrator, $P(F_0) = 1 - P(F_1)$. The results are as stated in Theorems I.5, and I.6 d and e. The formulas for attack protocol (b) seem to be impossible to simplify.

The interesting case is attack protocol (c). From now on we use h rather than y as the number of shots fired. From Lemma I.5,

$$P(F_h | DA_i) = \begin{cases} 1 - \frac{H}{i+1} & \text{if } h = \left\lfloor \frac{H}{i+1} \right\rfloor \\ \frac{H}{i+1} & \text{if } h = \left\lfloor \frac{H}{i+1} \right\rfloor + 1 \\ 0 & \text{for all other } h. \end{cases}$$

To evaluate

$$\sum_{t=0}^{\infty} P(F_h | DA_t) P(DA_t)$$

for given h , we must determine which t yield nonzero $P(F_h | DA_t)$. We define the following two series of sets, indexed by $h=0, \dots, H$, where t can assume only nonnegative integer values. Let

$$\mathbb{L}_1(h) = \left\{ t \mid h = \left\lfloor \frac{H}{t+1} \right\rfloor \right\}$$

$$\mathbb{L}_2(h) = \left\{ t \mid h = \left\lfloor \frac{H}{t+1} \right\rfloor + 1 \right\}.$$

For a fixed nonnegative integer h , $h = \left\lfloor \frac{H}{t+1} \right\rfloor$ (i.e., $t \in \mathbb{L}_1(h)$) if and only if

$$h \leq \frac{H}{t+1} < h+1.$$

Rearranging, if $h \geq 1$, this is equivalent to

$$\frac{H}{h+1} - 1 < t \leq \frac{H}{h} - 1.$$

Similarly, it can be shown that if $h \geq 2$, then $h = \left\lfloor \frac{H}{t+1} \right\rfloor + 1$, i.e., $t \in \mathbb{L}_2(h)$, if and only if

$$\frac{H}{h} - 1 < t \leq \frac{H}{h-1} - 1.$$

Therefore, for $h \geq 2$,

$$\mathbb{L}_1(h) = \left\{ t \mid \frac{H}{h+1} - 1 < t \leq \frac{H}{h} - 1 \right\}$$

$$\mathbb{L}_2(h) = \left\{ t \mid \frac{H}{h} - 1 < t \leq \frac{H}{h-1} - 1 \right\}.$$

The cases $h=0$ and $h=1$ must be treated specially. Note that $\left\lfloor \frac{H}{t+1} \right\rfloor = 0$ if and only

if $H < t+1$. Throughout, we are assuming that H is a positive integer. Thus $\left\lfloor \frac{H}{t+1} \right\rfloor = 0$

if and only if $t \geq H$, and

$$\mathbb{L}_1(0) = \{ H, H+1, H+2, \dots \}$$

By inspection, $\mathbb{L}_2(0)$ is the empty set, and $\mathbb{L}_2(1)=\mathbb{L}_1(0)$; $\mathbb{L}_1(1)$ can be found using the general formula.

It is evident that for any given h , $\mathbb{L}_1(h)$ and $\mathbb{L}_2(h)$ are disjoint, and that for $h_1 \neq h_2$, $\mathbb{L}_1(h_1)$ and $\mathbb{L}_1(h_2)$ are disjoint, as are $\mathbb{L}_2(h_1)$ and $\mathbb{L}_2(h_2)$. Also, $\mathbb{L}_2(h+1) = \mathbb{L}_1(h)$ for $h=0, \dots, H-1$. Some of the $\mathbb{L}_1(h)$ and $\mathbb{L}_2(h)$ may be empty.

We can find integer bounds on the ι . Let k be any integer and x any real number (possibly integer). It is clear that $k \leq x$ if and only if $k \leq \lfloor x \rfloor$, and it can be easily shown (treating the cases for integer and noninteger x separately) that

$$x < k \text{ if and only if } \lfloor x \rfloor + 1 \leq k.$$

Therefore the sets $\mathbb{L}_i(h)$ become

$$\mathbb{L}_1(0) = \{ H, H+1, H+2, \dots \}$$

$$\mathbb{L}_2(0) = \emptyset$$

$$\mathbb{L}_1(1) = \left\{ \iota \mid \left\lfloor \frac{H}{2} \right\rfloor \leq \iota \leq H-1 \right\}$$

$$\mathbb{L}_2(1) = \{ H, H+1, H+2, \dots \}$$

and if $H \geq 2$,

$$\mathbb{L}_1(h) = \left\{ \iota \mid \left\lfloor \frac{H}{h+1} \right\rfloor \leq \iota \leq \left\lfloor \frac{H}{h} \right\rfloor - 1 \right\} \quad h=2, \dots, H$$

$$\mathbb{L}_2(h) = \left\{ \iota \mid \left\lfloor \frac{H}{h} \right\rfloor \leq \iota \leq \left\lfloor \frac{H}{h-1} \right\rfloor - 1 \right\} \quad h=2, \dots, H$$

We can thus restate the conditions on $P(F_h|DA_1)$ as follows:

$$P(F_h|DA_1) = \begin{cases} 1 - \langle \frac{H}{i+1} \rangle & \text{if } i \in \mathbb{L}_1(h) \\ \langle \frac{H}{i+1} \rangle & \text{if } i \in \mathbb{L}_2(h) \\ 0 & \text{otherwise} . \end{cases}$$

Note that

$$\langle \frac{H}{i+1} \rangle = \frac{H}{i+1} - \left\lfloor \frac{H}{i+1} \right\rfloor$$

$$\left\lfloor \frac{H}{i+1} \right\rfloor = h \quad \text{for } i \in \mathbb{L}_1(h)$$

and

$$\left\lfloor \frac{H}{i+1} \right\rfloor = h-1 \quad \text{for } i \in \mathbb{L}_2(h) .$$

Thus the sum

$$\sum_{i=0}^{\infty} P(F_h|DA_1) P(DA_1)$$

used in the computation of the $P(F_h)$ becomes

$$\sum_{i \in \mathbb{L}_1(h)} (h+1 - \frac{H}{i+1}) P(DA_1) + \sum_{i \in \mathbb{L}_2(h)} (\frac{H}{i+1} - [h-1]) P(DA_1) .$$

Since $P(DA_1)$ is ω_i or ψ_{j1} , we can evaluate this sum using Lemma I.7, with the appropriate integer limits for $\mathbb{L}_1(h)$ and $\mathbb{L}_2(h)$ used as limits of summation. The set $\mathbb{L}_1(h)$ is empty if the corresponding expression in Lemma I.7 is negative, so we use the positive parts of such expressions. The formulas in Theorems I.5c and I.6c follow immediately.

4. The Heterogeneous Case Where $\alpha_j=0$

When the Poisson approximation to the binomial is used to determine the probabilities $P(DA_1)$, α_j , which is defined by

$$\alpha_j = \frac{w_j d_j}{L} (R_j - 1) + \sum_{\substack{i=1 \\ i \neq j}}^n \frac{w_i d_i}{L} R_i,$$

represents the mean number of penetrators, other than the specific one of type j under consideration, that a given barrier element can detect. If $\alpha_j=0$, then for $i \neq j$ either there are no penetrators of type i present or penetrators of type i cannot be detected (or both), and there is only the one specific penetrator of type j or penetrators of type j cannot be detected (or both).

Formally,

$$P(DA_1) = d_j \frac{e^{-\alpha_j} \alpha_j^1}{1!}, \quad 1=0,1,2, \dots$$

If $\alpha_j=0$, $P(DA_0)=d_j$ (which equals $P(D)$) and $P(DA_1)=0$ for all $1 \geq 1$. The general formulas of Chapter I still hold; the appropriate $P(F_h|DA_1)$ can be substituted in and the results follow immediately.

APPENDIX B

PROOFS OF RESULTS IN CHAPTER II

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PROOFS OF RESULTS IN CHAPTER II

Appendix B is organized similarly to Appendix A. The first part proves three lemmas used in the derivations. Lemmas II.1 and II.2, stated in Chapter II, Section C, concern the distribution of the events W_m . Lemma II.3 derives formulas for certain sums and weighted sums of the θ_{jl} (i.e., $P(DA_l)$). The second part of the appendix provides, for each theorem of Chapter II, Section E, proof details to complete the derivation steps outlined in Chapter II.

Unless specifically redefined, notation for all events, random variables, parameters, and so forth, is identical to that used in Chapter II.

A. THE LEMMAS

Lemma II.1. Let I_j be the integer part and f_j the fractional part of $w_j B/L$. Let the events V and W_m be as defined in Chapter II. Then, ignoring edge effects,

$$P(W_{I_j} | V) = \frac{I_j (1-f_j)}{(w_j B/L)}$$

$$P(W_{I_j+1} | V) = \frac{(I_j+1) f_j}{(w_j B/L)}$$

$$P(W_m | V) = 0 \quad \text{for all other } m.$$

Proof: Let the specific barrier element under consideration be denoted as barrier element β and the specific penetrator (of type j) as penetrator p . Note that β is an integer between one and B . Barrier element β 's position on the barrier line is, by Assumption 1)

$$x_\beta = \frac{L}{B} \left(\beta - \frac{1}{2} \right) .$$

Let \tilde{x} be the crossing point of penetrator p . By Assumption 4), if V occurs, \tilde{x} must lie within $w_j/2$ of x_β . We ignore edge effects by assuming that the distances of x_β from the edges of the barrier (points 0 and L) are at least $w_j/2$. (If $w_j \leq L/B$, this can easily be demonstrated.) Therefore, the interval of width w_j centered on point x_β does not overlap the edge of the barrier. The conditional distribution of \tilde{x} given V is therefore uniform on

$[x\beta - w_j/2, x\beta + w_j/2]$. We are interested in the total number of barrier elements (including β) that ρ can potentially detect, given V . This depends on the specific value x of ρ 's crossing point. Let $N_j(x)$ be this number of barrier elements. $N_j(x)$ is the cardinality of the set

$$\{b \mid |x_b - x| \leq w_j/2\},$$

where

$$x_b = \frac{L}{B} \left(b - \frac{1}{2}\right),$$

is, by Assumption 1), the location of barrier element b . In reality b can assume only integer values between one and B , inclusive, but we ignore edge effects by letting b assume any integer value. For any given x , the condition

$$|x_b - x| \leq w_j/2$$

is algebraically equivalent to

$$\frac{1}{2} + B \left(\frac{x}{L} - \frac{w_j}{2L} \right) \leq b \leq \frac{1}{2} + B \left(\frac{x}{L} + \frac{w_j}{2L} \right).$$

Let us denote the left endpoint of this interval by $b_0(x)$; the right endpoint is then $b_0(x) + (w_j B/L)$. If $b_0(x)$ is not an integer (and given that x is uniform, $b_0(\tilde{x})$ will be integer with probability zero), then $N_j(x)$, the number of integer points b in the interval

$$\left[b_0(x), b_0(x) + w_j B/L \right]$$

can be shown by the same methods used in the proof of Lemma I.3 to be

$$N_j(x) = I_j + \lfloor \langle b_0(x) \rangle + f_j \rfloor.$$

It is clear that

$$\lfloor \langle b_0(x) \rangle + f_j \rfloor = \begin{cases} 0 & \text{if } \langle b_0(x) \rangle < 1 - f_j \\ 1 & \text{if } \langle b_0(x) \rangle \geq 1 - f_j. \end{cases}$$

Thus, the random variable $N_j(\tilde{x})$ will assume only the values I_j or $I_j + 1$, and will equal $I_j + 1$ if and only if $\langle b_0(\tilde{x}) \rangle \geq 1 - f_j$. (W_m occurs if and only if $N_j(\tilde{x}) = m$.) We derive the probability of this event.

Given V , the crossing point \tilde{x} is uniformly distributed on $[x_{\beta-w_j/2}, x_{\beta+w_j/2}]$. Thus, $b_0(\tilde{x}) = \frac{1}{2} + B(\frac{x}{L} - \frac{w_j}{2L})$ is uniformly distributed on

$$\left[\frac{1}{2} + \frac{B}{L} (x_{\beta-w_j/2}), \frac{1}{2} + \frac{B}{L} x_{\beta} \right] .$$

Substituting in $\frac{L}{B} \left(\beta - \frac{1}{2} \right)$ for x_{β} and simplifying, this interval becomes

$$\left[\beta - \frac{Bw_j}{L}, \beta \right] .$$

Recall that β is a positive integer. The above interval is the disjoint union of the sequence of intervals

$$\begin{aligned} & \left[\beta - (w_j B/L), \beta - I_j \right] \\ & \left(\beta - I_j, \beta - (I_j - 1) \right] \\ & \left(\beta - (I_j - 1), \beta - (I_j - 2) \right] \\ & \quad \dots \\ & \left(\beta - 1, \beta \right] . \end{aligned}$$

The first interval has length f_j , the others have length unity.

Let the random variable Z be defined as the fractional part of $b_0(\tilde{x})$; i.e.,

$$Z = \langle b_0(\tilde{x}) \rangle .$$

We wish to determine

$$P(\tilde{z} \geq 1 - f_j).$$

If $b_0(\tilde{x})$ lies in one of the I_j unit intervals (which happens with probability $I_j/(I_j + f_j)$), \tilde{z} will be uniformly distributed on $[0,1]$ and

$$P(\tilde{z} \geq 1 - f_j) = f_j.$$

If $b_0(x)$ lies in the interval $[\beta - Bw_j/L, \beta - I_j]$ (which happens with probability $f_j/(I_j + f_j)$), then \tilde{z} , the fractional part of $b_0(\tilde{x})$, will certainly be at least $1 - f_j$. Therefore,

$$\begin{aligned} P(\tilde{z} \geq 1 - f_j) &= \frac{I_j}{I_j + f_j} f_j + \frac{f_j}{I_j + f_j} \\ &= \frac{f_j(I_j + 1)}{I_j + f_j} = \frac{f_j(I_j + 1)}{(w_j B/L)}. \end{aligned}$$

By our previous reasoning, this is the probability that $N_j(\tilde{x}) = I_j + 1$, and thus

$$P(N_j(\tilde{x}) = I_j) = 1 - \frac{f_j(I_j + 1)}{(w_j B/L)} = \frac{I_j(1 - f_j)}{(w_j B/L)},$$

which is the content of the lemma.

Lemma II.2. Let I_j , f_j , and W_m be as defined in Chapter II (and in Lemma II.1).

Then, ignoring edge effects:

$$\begin{aligned} P(W_{I_j}) &= 1 - f_j \\ P(W_{I_j+1}) &= f_j \\ P(W_m) &= 0 \quad \text{for all other } m. \end{aligned}$$

Proof. Again, call the specific penetrator under consideration penetrator p , assumed to be of type j . By Assumption 1), barrier element b is positioned at

$$x_b = \frac{L}{B} (b - \frac{1}{2})$$

and by Assumption 2), ρ 's crossing point \tilde{x} is uniformly distributed on $[0, L]$. By Assumption 4), for any given x , $N_j(x)$, the number of barrier elements that ρ can potentially detect, is the cardinality of the set

$$\{b \mid |x_b - x| \leq \frac{w_j}{2}\}.$$

We ignore edge effects by considering any integer b , even though b can, in fact, only be between 1 and B . If $w_j \leq L/B$ it is easily shown that for any $x \in [0, L]$, the above set will contain no integer points b outside of $[1, B]$. Exactly as in the proof of Lemma II.1, the condition $|x_b - x|$ is algebraically equivalent to

$$b_0(x) \leq b \leq b_0(x) + \frac{Bw_j}{L},$$

where

$$b_0(x) = \frac{1}{2} + \frac{B}{L} (x - \frac{w_j}{2}).$$

Also, as in Lemma II.1, the number of integer points b in $[b_0(x), b_0(x) + Bw_j/L]$ is $I_j + 1$ if and only if $\langle b_0(x) \rangle \geq 1 - f_j$. Now, however, \tilde{x} is uniformly distributed on $[0, L]$ (rather than $[x_\beta - w_j/2, x_\beta + w_j/2]$) and thus $b_0(\tilde{x})$ is uniformly distributed on

$$[\frac{1}{2} (1 - \frac{Bw_j}{L}), \frac{1}{2} (1 - \frac{Bw_j}{L}) + B].$$

Since B is a positive integer, this interval is composed of B intervals of length 1. As $b_0(\tilde{x})$ ranges over this interval, the fractional part of $b_0(\tilde{x})$ assumes in turn every value between 0 and 1, taking on each value B times. Thus, $\langle b_0(\tilde{x}) \rangle$ is uniformly distributed on $[0, 1]$ and

$$P(<b_0(\tilde{x})> \geq 1-f_j) = f_j .$$

But by previous reasoning, this is the probability that ρ can potentially detect $I_j + 1$ barrier elements.

Lemma II.3. Let I_j and f_j be as in Lemmas II.1 and II.2. As in Chapter II, Section E, let the probabilities $P(DA_t)$ be denoted by θ_{jt} , where

$$\begin{aligned} \theta_{jt} = & \frac{1}{B} I_j (1-f_j) d_j \binom{I_j-1}{t} d_j^t (1-d_j)^{I_j-1-t} \\ & + \frac{1}{B} (I_j+1) f_j d_j \binom{I_j}{t} d_j^t (1-d_j)^{I_j-1-t} \quad \text{for } t=0, \dots, I_j-1 \end{aligned}$$

and

$$\theta_{jI_j} = \frac{1}{B} (I_j+1) f_j d_j^{I_j+1}.$$

Then

a) For $0 \leq t_1 \leq t_2 \leq I_j-1$

$$\begin{aligned} \sum_{t=t_1}^{t_2} \theta_{jt} = & \frac{I_j (1-f_j) d_j}{B} [\mathbb{E}(t_2; I_j-1, d_j) - \mathbb{E}(t_1-1; I_j-1, d_j)] \\ & + \frac{(I_j+1) f_j d_j}{B} [\mathbb{E}(t_2; I_j, d_j) - \mathbb{E}(t_1-1; I_j, d_j)] . \end{aligned}$$

b) For $0 \leq t_1 \leq t_2 \leq I_j-1$

$$\begin{aligned} \sum_{t=t_1}^{t_2} \frac{1}{t+1} \theta_{jt} = & \frac{1}{B} (1-f_j) [\mathbb{E}(t_2+1; I_j, d_j) - \mathbb{E}(t_1; I_j, d_j)] \\ & + \frac{1}{B} f_j [\mathbb{E}(t_2+1; I_j+1, d_j) - \mathbb{E}(t_1; I_j+1, d_j)] . \end{aligned}$$

c) For $0 \leq t_1 \leq I_j$

$$\sum_{t=t_1}^{I_j} \theta_{jt} = \frac{I_j (1-f_j) d_j}{B} [1 - \mathbb{B}(t_1 - 1; I_j - 1, d_j)] \\ + \frac{(I_j + 1) f_j d_j}{B} [1 - \mathbb{B}(t_1 - 1; I_j, d_j)]$$

d) For $0 \leq t_1 \leq I_j$

$$\sum_{t=t_1}^{I_j} \frac{1}{t+1} \theta_{jt} = \frac{1}{B} (1-f_j) [1 - \mathbb{B}(t_1; I_j, d_j)] \\ + \frac{1}{B} f_j [1 - \mathbb{B}(t_1; I_j + 1, d_j)],$$

where the notation for binomial probabilities is as in Chapter I, Section G.

Proof. For simplicity in notation we suppress the subscript j on d_j , f_j , and I_j . Let

$$a_1 = \frac{1}{B} I (1-f) d$$

and

$$a_2 = \frac{1}{B} (I+1) f d .$$

Note that for $t = 0, \dots, I-1$

$$\theta_{jt} = a_1 \mathbb{b}(t; I-1, d) + a_2 \mathbb{b}(t; I, d)$$

and

$$\theta_{jI} = a_2 \mathbb{b}(I; I, d) .$$

Also, for $t = 0, \dots, I-1$

$$\begin{aligned} \frac{1}{t+1} b(t; I-1, d) &= \frac{1}{t+1} \frac{(I-1)!}{(I-1-t)! t!} d^t (1-d)^{I-1-t} \\ &= \frac{1}{Id} \frac{I!}{(I-[t+1])! (t+1)!} d^{t+1} (1-d)^{I-[t+1]} \\ &= \frac{1}{Id} b(t+1; I, d) . \end{aligned}$$

$$\text{Thus, } \frac{1}{t+1} a_1 b(t; I-1, d) = \frac{1}{B} (1-f) b(t+1; I, d) .$$

For $t = 0, \dots, I$

$$\begin{aligned} \frac{1}{t+1} b(t; I, d) &= \frac{1}{t+1} \frac{I!}{(I-t)! t!} d^t (1-d)^{I-t} \\ &= \frac{1}{(I+1)d} \frac{(I+1)!}{([I+1]-[t+1])! (t+1)!} d^{t+1} (1-d)^{I-t} \\ &= \frac{1}{(I+1)d} b(t+1; I+1, d) . \end{aligned}$$

$$\text{Thus, } \frac{1}{t+1} a_2 b(t; I, d) = \frac{1}{B} f b(t+1; I+1, d) .$$

Thus, for $0 \leq t_1 \leq t_2 \leq I-1$

$$\sum_{t=t_1}^{t_2} \theta_{jt} = \sum_{t=t_1}^{t_2} [a_1 b(t; I-1, d) + a_2 b(t; I, d)]$$

and

$$\sum_{t=t_1}^{t_2} \frac{1}{t+1} \theta_{jt} = \sum_{t=t_1}^{t_2} \frac{1}{B} (1-f) b(t+1; I, d) + \sum_{t=t_1}^{t_2} \frac{1}{B} f b(t+1; I+1, d) .$$

Formulas a) and b) follow forthwith from the definition of the cumulative binomial.

Assume that $t_1 \leq I-1$. Then

$$\sum_{t=t_1}^I \theta_{jt} = \sum_{t=t_1}^{I-1} a_1 b(t; I-1, d) + \sum_{t=t_1}^I a_2 b(t; I, d)$$

and

$$\sum_{i=t_1}^I \frac{1}{i+1} \theta_{ji} = \sum_{i=t_1}^{I-1} \frac{1}{B} (1-f) b(i+1; I, d) \\ + \sum_{i=t_1}^{I_j} \frac{1}{B} f b(i+1; I+1, d) .$$

Formulas c) and d) follow forthwith.

If $t_1 = I$, then

$$\sum_{i=t_1}^I \theta_{ji} = a_2 \theta_{jI} = a_2 b(I; I, d)$$

If $t_1 = I$ is substituted in formula c), the first term becomes zero, and the second term is

$$a_2 [1 - B(I-1; I, d)] = a_2 b(I; I, d) .$$

In formula d), substituting I for t_1 yields

$$0 + \frac{1}{B} f [1 - B(I; I+1, d)] ,$$

which equals

$$\frac{1}{B} f b(I+1; I+1, d) ,$$

or

$$\frac{1}{I+1} \theta_{jI} .$$

Thus formulas c) and d) are true for all i , between 0 and I , inclusive.

B . PROOF DETAILS OF THEOREMS

The discussion in Section C of Chapter II has presented the overall steps in the derivations of the desired quantities. First, quantities q_j , e_j , and ϕ_{hj} are derived for one penetrator and one barrier element. These are then combined as explained in Chapter II, Section C.2. As derived in Section C.1, the formulas for q_j , e_j , and ϕ_{hj} are

$$\begin{aligned}
q_j &= \sum_{h=0}^{H_j} \sum_{t=0}^{I_j} P(K|F_h) P(F_h|DA_t) P(DA_t) \\
&= \sum_{t=0}^{I_j} P(K|DA_t) P(DA_t) \\
e_j &= \sum_{h=0}^{H_j} \sum_{t=0}^{I_j} h P(F_h|DA_t) P(DA_t) \\
&= \sum_{t=0}^{I_j} E(\bar{h}|DA_t) P(DA_t)
\end{aligned}$$

$$\phi_{0j} = 1 - \frac{w_j d_j}{L} + \sum_{t=0}^{I_j} P(F_0|DA_t) P(DA_t)$$

and

$$\phi_{hj} = \sum_{t=0}^{I_j} P(F_h|DA_t) P(DA_t), \quad h = 1, \dots, H_j.$$

The $P(F_h|DA_t)$ are given by Lemma I.5 in Appendix A, the $P(DA_t)$ are simply the θ_{jt} of Lemma II.3, and the $P(K|DA_t)$ and $E(\bar{h}|DA_t)$ are derived in Lemma I.6. Examining these quantities, we see that most of the $P(K|DA_t)$, $E(\bar{h}|DA_t)$, and $P(F_h|DA_t)$ are either constants with respect to t or involve a denominator of $t+1$. Lemma II.3 can then be used to evaluate the above indicated sums. In attack protocols (b) and (c), we have not been able to derive closed forms for q_j and ϕ_{hj} , except as noted in Theorems II.3b and c.

The interesting case is attack protocol (d), where $P(F_1|DA_t) = E(\bar{h}|DA_t) = \min(\frac{H_j}{t+1}, 1)$, and $P(K|DA_t) = k_j \min(\frac{H_j}{t+1}, 1)$. To evaluate

$$e_j = \sum_{t=0}^{I_j} \min\left(\frac{H_j}{t+1}, 1\right) \theta_{jt},$$

we first note that the minimum is less than 1 if and only if $i \geq H_j$. If $H_j \geq I_j + 1$, for all values of i with positive θ_{ji} , the $E(\tilde{h}|DA_1)$ term will equal 1. Lemma II.3 formula c) then yields (treating $B(-1; n, p) = 0$)

$$\begin{aligned} e_j &= \sum_{i=0}^{H_j} \theta_{ji} = \frac{1}{B} \left[I_j (1-f_j) d_j + (I_j+1) f_j d_j \right] \\ &= \frac{1}{B} d_j \frac{w_j B}{L} \\ &= \frac{w_j d_j}{L} \end{aligned}$$

If $H_j \leq I_j$, the formula for e_j becomes

$$e_j = \sum_{i=0}^{H_j-1} \theta_{ji} + \sum_{i=H_j}^{I_j} \frac{H_j}{i+1} \theta_{ji}$$

These sums can be evaluated by formulas a) and d) of Lemma II.3, respectively. When multiplied by k_j , e_j becomes q_j , which is presented in the statement of Theorem II.1d.